Small-time asymptotics and numerical simulation of stopped Lévy processes

José E. Figueroa-Lopez and **Peter Tankov**

7th International Conference on Lévy Processes:
Theory and Applications

July 17, 2013

Outline

1. Small-time asymptotics for stopped Lévy processes
2. Bridge-based simulation algorithms
3. Adaptive simulation of stopped Lévy processes with bias control
4. Numerical illustrations

Outline

1. Small-time asymptotics for stopped Lévy processes
2. Bridge-based simulation algorithms
3. Adaptive simulation of stopped Lévy processes with bias control
4. Numerical illustrations

Small-time asymptotics of jump processes

Important for simulation, statistics, option pricing

Classical result by Léandre (1987) :

Proposition

Let X be a Lévy process with triplet (σ^2, ν, μ)

Assume ν admits a density $s : \mathbb{R} \to (0, \infty)$ and that the law of X_t admits a density $f_t(x)$ for all $t > 0$.

Then,

$$\lim_{t \to 0} \frac{1}{t} f_t(x) = s(x), \qquad (x \neq 0).$$

Stopped Lévy processes

We characterize small-time asymptotics of path-dependent functionals such as stopped Lévy processes

$$\mathbb{E}\left[\varphi(X_\tau)\mathbf{1}_{\tau\leq t}\mathbf{1}_{X_t\in I}\right]$$

$\tau := \inf\{u \geq 0 : X_u \notin (a, b)\}$ with $-\infty \leq a < 0 < b \leq \infty$ and $I \in (a, b)$.

Stopped Lévy processes

We characterize small-time asymptotics of path-dependent functionals such as stopped Lévy processes

$$\mathbb{E}\left[\varphi(X_\tau)\mathbf{1}_{\tau\le t}\mathbf{1}_{X_t\in I}\right]$$

$\tau := \inf\{u \ge 0 : X_u \notin (a,b)\}$ with $-\infty \le a < 0 < b \le \infty$ and $I \in (a,b)$.

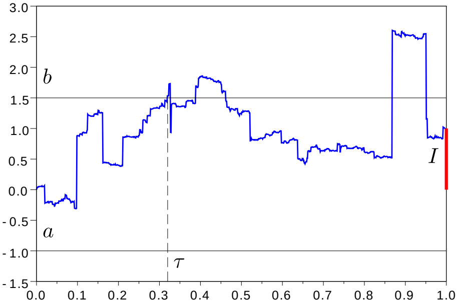
Examples :

Exit probability

$$\mathbb{P}\left[\tau \le t, X_t \in I\right]$$

Stopped Lévy bridge

$$\mathbb{E}\left[\varphi(X_\tau)\mathbf{1}_{\tau\le t}|X_t = y\right], \ y \in (a,b)$$



Main motivation :
Monte Carlo evaluation of functionals such as

$$\mathbb{E}[F(X_T)\mathbf{1}_{\tau > T}], \qquad \tau = \inf\{t \geq 0 : X_t \notin (a, b)\}$$

Main motivation :
Monte Carlo evaluation of functionals such as

$$\mathbb{E}[F(X_T)\mathbf{1}_{\tau>T}], \qquad \tau = \inf\{t \geq 0 : X_t \notin (a,b)\}$$

$\Rightarrow$ Correct the discretization bias with asymptotic exit probability

$\Rightarrow$ Zoom in on parts of trajectory which are close to the barrier using adaptive bridge simulation

Main result

Theorem

$$\mathbb{E}\left(\varphi(X_\tau)\mathbf{1}_{\tau\le t}\mathbf{1}_{X_t\in I}\right) = \frac{t^2}{2}\int_I du \int\limits_{(a,b)^c} dv\,\varphi(v)\,s(v)s(u-v) + O(t^{\frac{5}{2}})$$

when $t \to 0$ for $I \subset (a, b)$.

✓ We provide a precise computable bound for the remainder

Main result

Theorem

$$\mathbb{E}\left(\varphi(X_\tau)\mathbf{1}_{\tau\leq t}\mathbf{1}_{X_t\in I}\right) = \frac{t^2}{2}\int_I du \int_{(a,b)^c} dv\,\varphi(v)\,s(v)s(u-v) + O(t^{\frac{5}{2}})$$

when $t \to 0$ *for* $I \subset (a,b)$.

✓ We provide a precise computable bound for the remainder

Intuition :

If, in small time, a Lévy process goes out of (a,b) and then comes back to $I \subset (a,b)$, this essentially happens with two large jumps :

– the first jump takes the process out of (a,b),

– the second jump brings it back to $I \subset (a,b)$.

Intuition from finite intensity

Taking $\varphi = 1$, yields the exit probability :

$$\mathbb{P}(\tau \le t, X_t \in I) = \frac{t^2}{2}\int_I du \int_{(a,b)^c} dv\, s(v)s(u-v) + O(t^{\frac{5}{2}})$$

For a compound Poisson process $X_t = \sum_{i=1}^{N_t} Y_i$ with jump intensity λ and jump size density p,

$$\begin{aligned}
\mathbb{P}(\tau \le t, X_t \in I) &= \mathbb{P}(\exists u \in [0,t]:\ X_u \notin (a,b), X_t \in I)\\
&\approx \frac{\lambda^2 t^2}{2} \times \mathbb{P}(Y_1 \notin (a,b), Y_1 + Y_2 \in I)\\
&= \frac{\lambda^2 t^2}{2}\int_I du \int_{(a,b)^c} p(v)p(u-v)\, dv\\
&= \frac{t^2}{2}\int_I du \int_{(a,b)^c} s(v)s(u-v)\, dv.
\end{aligned}$$

Lévy bridges

Exit probability of the Lévy bridge :

$$\begin{aligned}\mathbb{P}\left(\tau \le t \middle| X_t = y\right) &= \lim_{\delta\to 0} \frac{\mathbb{P}\left(\tau \le t, X_t \in (y-\delta, y+\delta)]\right)}{\mathbb{P}\left(X_t \in (y-\delta, y+\delta)]\right)}\\ &= \frac{t^2}{2}\int_{(a,b)^c} \frac{s(v)s(y-v)}{f_t(y)}\,dv + O(t^{\frac{3}{2}})\\ &= \frac{t}{2}\int_{(a,b)^c} \frac{s(v)s(y-v)}{s(y)}\,dv + O(t^{\frac{3}{2}}) \qquad (y \neq 0).\end{aligned}$$

Lévy bridges

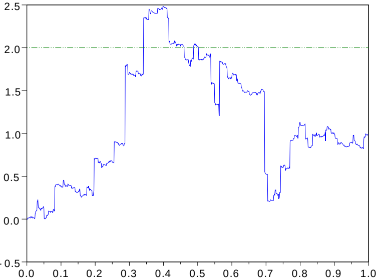
Exit probability of the Lévy bridge :

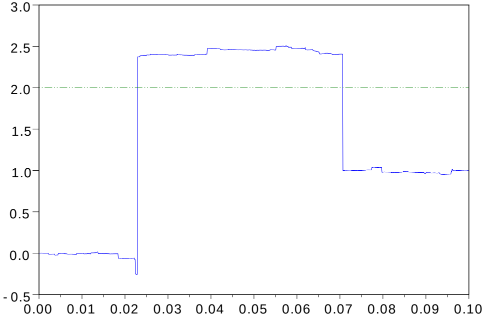
$$\begin{aligned}\mathbb{P}\left(\tau \leq t \middle| X_t = y\right) &= \lim_{\delta \to 0} \frac{\mathbb{P}\left(\tau \leq t, X_t \in (y-\delta, y+\delta)]\right)}{\mathbb{P}\left(X_t \in (y-\delta, y+\delta)]\right)} \\ &= \frac{t^2}{2} \int_{(a,b)^c} \frac{s(v)s(y-v)}{f_t(y)}\, dv + O(t^{\frac{3}{2}}) \\ &= \frac{t}{2} \int_{(a,b)^c} \frac{s(v)s(y-v)}{s(y)}\, dv + O(t^{\frac{3}{2}}) \qquad (y \neq 0).\end{aligned}$$

Remarkable asymptotic identity :

$$\mathbb{P}\left(\tau \leq t \middle| X_t = y\right) \sim 2\mathbb{P}\left(X_{\frac{t}{2}} \notin (a,b) \middle| X_t = y\right), \qquad t \to 0.$$

“Spaghetti theorem”

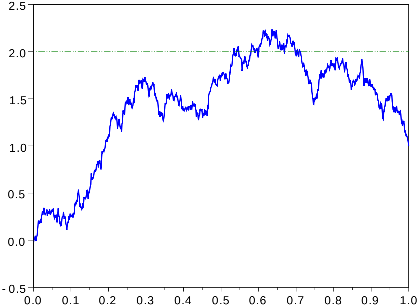


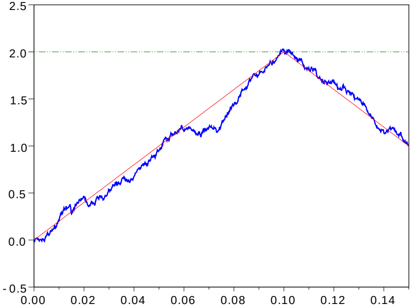


Left : Cauchy bridge conditioned to cross level $x = 2$, on $[0, 1]$.

Right : Cauchy bridge conditioned to cross level $x = 2$, on $[0, 0.1]$.

Comparison to the diffusion case





Left : Brownian bridge conditioned to cross level $x = 2$, on $[0, 1]$.

Right : Brownian bridge conditioned to cross level $x = 2$, on $[0, 0.15]$.

Outline

Traditional Monte Carlo method

Goal : evaluate the expectation of a path-dependent functional $\mathbb{E}[\mathcal{H}]$ of a Lévy process

1. Discretize time : $0 = t_0 < \cdots < t_n = T$
2. Generate the sample : $X_{t_1}, \ldots, X_{t_n}$
   (from the law of increments $\Delta_i^n X := X_{t_i} - X_{t_{i-1}}$) ;
3. Compute a discretized approximation $\widetilde{\mathcal{H}}$ of the functional $\mathcal{H}$
4. Repeat (1)-(3) to generate m copies of $\widetilde{\mathcal{H}}$: $\widetilde{\mathcal{H}}_1, \ldots, \widetilde{\mathcal{H}}_m$.
5. Compute the MC estimate $\frac{1}{m}\sum_{i=1}^{m} \widetilde{\mathcal{H}}_i$.

Bridge Monte Carlo

1. Simulate the terminal value X_T from the law of the increment.
2. Simulate intermediate points using the bridge law

$$f_t^{br}(\cdot|t_1, x_1, t_2, x_2) = Law(X_t|X_{t_1} = x_1, X_{t_2} = x_2)$$

Bridge Monte Carlo

1. Simulate the terminal value X_T from the law of the increment.
2. Simulate intermediate points using the bridge law

$$f_t^{br}(\cdot|t_1, x_1, t_2, x_2) = Law(X_t|X_{t_1} = x_1, X_{t_2} = x_2)$$

$X_{\frac{T}{2}}$ is simulated from $f_{\frac{T}{2}}^{br}(\cdot|0, 0, T, X_T)$
$X_{\frac{T}{4}}$ is simulated from $f_{\frac{T}{4}}^{br}(\cdot|0, 0, \frac{T}{2}, X_{\frac{T}{2}})$
$X_{\frac{3}{4}T}$ is simulated from $f_{\frac{3}{4}T}^{br}(\cdot|\frac{T}{2}, X_{\frac{T}{2}}, T, X_T)$ etc.

Bridge Monte Carlo

1. Simulate the terminal value X_T from the law of the increment.
2. Simulate intermediate points using the bridge law

$$f_t^{br}(\cdot|t_1, x_1, t_2, x_2) = Law(X_t|X_{t_1} = x_1, X_{t_2} = x_2)$$

$X_{\frac{T}{2}}$ is simulated from $f^{br}_{\frac{T}{2}}(\cdot|0, 0, T, X_T)$
$X_{\frac{T}{4}}$ is simulated from $f^{br}_{\frac{T}{4}}(\cdot|0, 0, \frac{T}{2}, X_{\frac{T}{2}})$
$X_{\frac{3}{4}T}$ is simulated from $f^{br}_{\frac{3}{4}T}(\cdot|\frac{T}{2}, X_{\frac{T}{2}}, T, X_T)$ etc.

Advantages :

- The trajectory can be adaptively refined only where necessary
- Variance reduction methods are easy to design by replacing the density of X_T with an importance sampling distribution

Drawbacks :

- Sampling from the bridge law necessary — we propose a rejection algorithm for LP with unimodal densities

Outline

Introduction

1. Goal : Evaluate

$$C(0; T, F) := \mathbb{E}[F(X_T)\mathbf{1}_{\tau > T}], \qquad \text{where,}$$
$$\tau := \inf\{u \geq 0 : X_u \notin (a, b)\}, \quad T > 0.$$

2. Motivation :
   - Barrier options : If X_t models the log-return process of a risky asset, then $C(0; T, F)$ could represent the time-0 price of a European Barrier option.
   - Convertible or defaultable bonds : In structural credit risk models, the default or conversion of a bond is often specified as the passage of an underlying risky factor $\{X_t\}$ across a level.
   - Superdiffusion in confined domains. In physics or chemistry, Lévy processes are used as models of certain superdiffusive phenomena.

Traditional Monte Carlo for stopped processes

1. Discretize time : $0 = t_0 < \cdots < t_n = T$
2. Generate the sample : $X_{t_1}, \ldots, X_{t_n}$
   (from the law of increments $\Delta_i^n X := X_{t_i} - X_{t_{i-1}}$) ;
3. Approximate the exit time : $\tilde{\tau}_n := \min\{t_k : X_{t_k} \notin (a, b)\}$;
4. Compute approximate payoff :
   $\mathcal{H} := F(X_T)\mathbf{1}_{\tau > T} \approx \widetilde{\mathcal{H}} := F(X_{t_n})\mathbf{1}_{\tilde{\tau}_n > T}$.
5. Repeat (1)-(4) to generate m copies of approx. payoffs :
   $\widetilde{\mathcal{H}}_1, \ldots, \widetilde{\mathcal{H}}_m$.
6. MC estimate : $\widetilde{C}(0; T, F) := \frac{1}{m}\sum_{i=1}^m \widetilde{\mathcal{H}}_i$.

Sequential Monte Carlo for stopped processes

Drawback of sequential MC for stopped processes :

> *Highly biased due to the possibility of exiting the interval* (a, b) *between sampling observations.*

The bias is of order of $\frac{1}{\sqrt{n}}$ for diffusions (Asmussen, Glynn and Pitman ; 1995), (Broadie, Glasserman and Kou, 1997) and diffusions with finite intensity jumps (Dia and Lamberton, 2010) but unknown for general Lévy processes.

Improved MC Method (Baldi, 1995)

1. Suppose there is a way to compute the exit probability :

$$p(x, y, t) := \mathbb{P}\left(\exists u \in [s, s+t] : \ X_u \notin (a, b) \middle| X_s = x, X_{s+t} = y\right).$$

Improved MC Method (Baldi, 1995)

1. Suppose there is a way to compute the exit probability :

$$p(x,y,t) := \mathbb{P}\left(\exists u \in [s,s+t]:\ X_u \notin (a,b) \middle| X_s = x, X_{s+t} = y\right).$$

2. By the Markov Property,

$$\begin{aligned}\mathbb{E}[F(X_T)\mathbf{1}_{\tau>T}] &= \mathbb{E}\Big[F(X_T)\prod_{i=0}^{n-1}\mathbf{1}_{\{\forall u\in(t_{i-1},t_i]:X_u\in(a,b)\}}\Big]\\ &= \mathbb{E}\Big[F(X_T)\prod_{i=0}^{n-1}\mathbb{E}\big[\mathbf{1}_{\{\forall u\in(t_{i-1},t_i]:X_u\in(a,b)\}}\big|X_{t_{i-1}},X_{t_i}\big]\Big]\\ &= \mathbb{E}\left[F(X_T)\prod_{i=0}^{n-1}\big(1-p(X_{t_i},X_{t_{i+1}},t_{i+1}-t_i)\big)\right].\end{aligned}$$

Improved MC Method. Cont...

1. Generate the sample : $X_{t_1}, \ldots, X_{t_n}$
2. Compute the expected pay-off conditional on the skeleton $\widetilde{\mathcal{H}} := F(X_{t_n}) \prod_{i=0}^{n-1} \left(1 - p(X_{t_i}, X_{t_{i+1}}, t_{i+1} - t_i)\right)$.
3. Repeat (1)-(2) to generate m copies : $\widetilde{\mathcal{H}}_1, \ldots, \widetilde{\mathcal{H}}_m$.
4. MC estimate : $\widetilde{C}(0; T, F) := \frac{1}{m} \sum_{i=1}^{m} \widetilde{\mathcal{H}}_i$.

Improved MC Method. Cont...

1. Generate the sample : $X_{t_1}, \dots, X_{t_n}$
2. Compute the expected pay-off conditional on the skeleton $\widetilde{\mathcal{H}} := F(X_{t_n}) \prod_{i=0}^{n-1} \left(1 - p(X_{t_i}, X_{t_{i+1}}, t_{i+1} - t_i)\right)$.
3. Repeat (1)-(2) to generate m copies : $\widetilde{\mathcal{H}}_1, \dots, \widetilde{\mathcal{H}}_m$.
4. MC estimate : $\widetilde{C}(0; T, F) := \frac{1}{m} \sum_{i=1}^{m} \widetilde{\mathcal{H}}_i$.

$p(x, y, t)$ is not typically known in closed form or easy to compute ;

Improved MC Method. Cont...

1. Generate the sample : $X_{t_1}, \dots, X_{t_n}$
2. Compute the expected pay-off conditional on the skeleton $\widetilde{\mathcal{H}} := F(X_{t_n}) \prod_{i=0}^{n-1} \left(1 - p(X_{t_i}, X_{t_{i+1}}, t_{i+1} - t_i)\right)$.
3. Repeat (1)-(2) to generate m copies : $\widetilde{\mathcal{H}}_1, \dots, \widetilde{\mathcal{H}}_m$.
4. MC estimate : $\widetilde{C}(0; T, F) := \frac{1}{m} \sum_{i=1}^{m} \widetilde{\mathcal{H}}_i$.

$p(x, y, t)$ is not typically known in closed form or easy to compute ;

Proposed solution : Short-time approximation :

$$p(x, y, t) := \mathbb{P}\left(\exists u \in [0, t] :\ X_u \notin (a - x, b - x) \middle| X_0 = 0, X_t = y - x\right)$$
$$\approx \frac{t^2}{2} \int_{(a-x, b-x)^c} \frac{s(v)s(y - x - v)}{f_t(y - x)}\, dv =: \tilde{p}(x, y, t).$$

Controlling the bias

Approximate the conditional expected pay-off :

$$\widetilde{\mathcal{H}} := F(X_{t_n}) \prod_{i=0}^{n-1} \left(1 - \tilde{p}(X_{t_i}, X_{t_{i+1}}, t_{i+1} - t_i)\right).$$

Controlling the bias

Approximate the conditional expected pay-off :

$$\widetilde{\mathcal{H}} := F(X_{t_n}) \prod_{i=0}^{n-1} \left(1 - \tilde{p}(X_{t_i}, X_{t_{i+1}}, t_{i+1} - t_i)\right).$$

Assuming an estimate $e_p(x, y, t)$ of the error is available :

$$|p(x, y, t) - \tilde{p}(x, y, t)| \leq e_p(x, y, t),$$

one can control the global bias if for desired tolerance γ,

$$e_p(X_{t_i}, X_{t_{i+1}}, t_{i+1} - t_i) \leq \gamma(t_{i+1} - t_i) \quad \text{for all } i.$$

Controlling the bias

Approximate the conditional expected pay-off :

$$\widetilde{\mathcal{H}} := F(X_{t_n}) \prod_{i=0}^{n-1} \left(1 - \tilde{p}(X_{t_i}, X_{t_{i+1}}, t_{i+1} - t_i)\right).$$

Assuming an estimate $e_p(x, y, t)$ of the error is available :

$$|p(x, y, t) - \tilde{p}(x, y, t)| \leq e_p(x, y, t),$$

one can control the global bias if for desired tolerance γ,

$$e_p(X_{t_i}, X_{t_{i+1}}, t_{i+1} - t_i) \leq \gamma(t_{i+1} - t_i) \quad \text{for all } i.$$

Requires adaptive discretization
because $e_p(x, y, t)$ increases when x or y are close to the barrier

Adaptive bridge-based method with controlled bias

Replace the deterministic dates $(t_0, \ldots, t_n)$ by a random discretization

Adaptive bridge-based method with controlled bias

Replace the deterministic dates $(t_0, \ldots, t_n)$ by a random discretization

Refine the discretization adaptively using the bridge law when $e_p(x, y, t)$ is big.

Adaptive bridge-based method with controlled bias

Replace the deterministic dates $(t_0, \dots, t_n)$ by a random discretization

Refine the discretization adaptively using the bridge law when $e_p(x, y, t)$ is big.

The procedure returns a random skeleton $\mathcal{X} = \{(T_i, X_{T_i})\}_{i=0}^N$ with

$$e_p(X_{T_{(i)}}, X_{T_{(i+1)}}, T_{(i+1)} - T_{(i)}) \leq \gamma(T_{(i+1)} - T_{(i)}),$$

for all i, where $0 = T_{(0)} < \cdots < T_{(N)}$ are the order “statistics” of $\{T_0, \dots, T_N\}$

Adaptive Bridge-based method with controlled bias

Returns $\tilde{N}(\mathcal{X}) := \prod_{i=0}^{N-1}\left(1 - \tilde{p}(X_{T_{(i)}}, X_{T_{(i+1)}}, T_{(i+1)} - T_{(i)})\right)$.

FUNCTION N(parameters: x, y, ΔT)

IF $x \notin (a,b)$ OR $y \notin (a,b)$ THEN RETURN 0

IF $e_p(x,y,\Delta T) \leq \gamma \Delta T$ THEN RETURN $1 - \tilde{p}(x,y,\Delta T)$

ELSE

Sample $\hat{X}$ from the bridge distribution

$$X_{\frac{\Delta T}{2}} | X_0 = x, X_{\Delta T} = y$$

RETURN N$(x, \hat{X}, \Delta T/2) \times$N$(\hat{X}, y, \Delta T/2)$

END IF

Convergence of the algorithm

Proposition

Assume that the Lévy process satisfies one of the conditions :

1. *X has infinite variation and does not hit points, that is,*

$$\int_{\mathbb{R}} \Re\left(\frac{1}{1-\psi(u)}\right) du = \infty, \quad \psi(u) = \log \mathbb{E}[e^{iuX_1}].$$

   (examples : Cauchy, Normal Inverse Gaussian)

2. *X has finite variation (example : Variance Gamma)*

then the adaptive algorithm terminates in finite time a.s.

In other cases, a slight modification of the algorithm allows to ensure the convergence while still controlling the bias.

Outline

Implementation issues

For the implementation, the following computations are needed :

Implementation issues

For the implementation, the following computations are needed :

- Simulation of the increments of X_t for arbitrary t ;

Implementation issues

For the implementation, the following computations are needed :

- Simulation of the increments of X_t for arbitrary t ;
- Evaluation of the density f_t of X_t for arbitrary t ;

Implementation issues

For the implementation, the following computations are needed :

- Simulation of the increments of X_t for arbitrary t ;
- Evaluation of the density f_t of X_t for arbitrary t ;
- Evaluation of the error bound $e_p(0, y, t)$
  ✓ explicit formula in terms of the Lévy density

Implementation issues

For the implementation, the following computations are needed :

- Simulation of the increments of X_t for arbitrary t;
- Evaluation of the density f_t of X_t for arbitrary t;
- Evaluation of the error bound $e_p(0, y, t)$
  ✓ explicit formula in terms of the Lévy density
- To approximate the exit probability $p(x, y, t)$
  – evaluation of the "incomplete convolution"

$$\mathcal{C}(b, y) := \int_b^\infty s(v)s(y - v)dv;$$

Implementation issues

For the implementation, the following computations are needed :

- Simulation of the increments of X_t for arbitrary t ;
- Evaluation of the density f_t of X_t for arbitrary t ;
- Evaluation of the error bound $e_p(0, y, t)$
  ✓ explicit formula in terms of the Lévy density
- To approximate the exit probability $p(x, y, t)$
  – evaluation of the "incomplete convolution"

$$\mathcal{C}(b, y) := \int_b^\infty s(v)s(y-v)dv;$$

Alternatively, one could use the asymptotic identity

$$\mathbb{P}[\tau \le t | X_0 = x, X_t = y] \sim 2\mathbb{P}[X_{t/2} \notin (a, b) | X_0 = x, X_t = y].$$

Example : Cauchy process

Lévy process with Lévy density $s(x) = \frac{c}{|x|^2}$,

$$\mathcal{C}(b, y) = \frac{c^2}{b^3} \left\{ 1 + \frac{b}{y} + \frac{2b^2}{y^2} + \frac{y}{b-y} + \frac{2b^3}{y^3} \log\left(1 - \frac{y}{b}\right) \right\}.$$

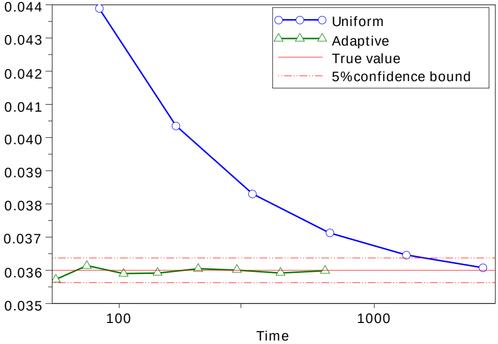
The probability density is given by

$$f_t(x) = \frac{ct}{(\pi ct)^2 + x^2}.$$

Approximate the law of the maximum using results of Darling (1956) :

$$\mathbb{P}[\sup_{0 \le s \le 1} X_s \le x] = \frac{2}{\pi}\sqrt{\frac{x}{\pi c}} - \frac{2}{3\pi^2}\left(\frac{x}{\pi c}\right)^{\frac{3}{2}} \left\{ \log \frac{x}{\pi c} - \frac{5}{3} \right\} + O\left(x^{\frac{5}{2}}\right).$$

Example : Cauchy process



$\mathbb{P}[\sup_{0\leq s\leq 1} X_s \leq 10^{-2}]$ computed with uniform and adaptive discretization, as function of computing time (sec.) for 10^6 paths

Uniform discretization : we use from 256 to 16384 points

Adaptive algorithm : γ ranges from 9 to 9×10^{-3}

Conclusions and extensions

Main results :

- Asymptotics for stopped Lévy processes / bridges with precise error bounds ;
- Bridge simulation method for general Lévy processes ;
- Monte Carlo simulation of stopped Lévy processes with controlled bias.

Extensions :

- Improved asymptotic approximations close to the boundary ;
- Simulation of stopping times and overshoots ;
- Multidimensional Lévy processes in confined domains ;
- General Markov jump processes.

Reference : J. E. Figueroa-Lopez and P. Tankov, *Small-time asymptotics of stopped Lévy bridges and simulation schemes with controlled bias* , Arxiv : 1203.2355, to appear in *Bernoulli*