

# On entrance boundaries and a generalised notion of an interlacement

Steffen Dereich (joint work with Leif Döring)

WWU Münster

<http://wwwmath.uni-muenster.de/statistik/dereich/>

7th International Conference on Lévy Processes:  
Theory and Applications

Wroclaw, 15/07/2013

# Motivation

**Central Question:** How to start a Markov process from a boundary point?

**Lévy processes/pssMps:** Bertoin, Caballero, Chaumont, Döring, Fitzsimmons, Kyprianou, Lamperti, Pardo, Rivero, Savov, Yor, ..

# Motivation

**Central Question:** How to start a Markov process from a boundary point?

**Lévy processes/pssMps:** Bertoin, Caballero, Chaumont, Döring, Fitzsimmons, Kyprianou, Lamperti, Pardo, Rivero, Savov, Yor, ..

**Coalescent:** Aldous, Limic, Pitman, ..

**Galton-Watson process:** Alsmeyer, Athreya, Ney, Rösler

**Lamplighter walk:** Woess, ..

# Motivation

**Central Question:** How to start a Markov process from a boundary point?

**Lévy processes/pssMps:** Bertoin, Caballero, Chaumont, Döring, Fitzsimmons, Kyprianou, Lamperti, Pardo, Rivero, Savov, Yor, ..

**Coalescent:** Aldous, Limic, Pitman, ..

**Galton-Watson process:** Alsmeyer, Athreya, Ney, Rösler

**Lamplighter walk:** Woess, ..

**Related to:** Potential analysis, Martin boundaries, ...

**General theory:** Doob, Dynkin, Feller, Hunt, Kunita, Martin, Pinsky, Watanabe, ..

**Martin kernels, etc:** Bogdan, Byczkowski, Damek, Vondracek, ..

# The Markov process

## Setting:

- ▶  $S$  locally compact Polish space (non-compact), the **state space**
- ▶  $\bar{S} = S \cup \{\partial\}$  one-point compactification
- ▶  $(X_t)_{t \geq 0}$  càdlàg Feller **Markov process**; under  $P^x$  started in  $x \in S$

## Running examples:

- E1  $S = (0, \infty)$ ,  $(X_t)$  Brownian motion killed when hitting the boundary
- E2  $S = \mathbb{R}^d$ ,  $(X_t)$  Brownian motion
- E3  $S = \mathbb{R}$ ,  $(X_t)$  subordinator

# The Markov process

## Setting:

- ▶  $S$  locally compact Polish space (non-compact), the **state space**
- ▶  $\bar{S} = S \cup \{\partial\}$  one-point compactification
- ▶  $(X_t)_{t \geq 0}$  càdlàg Feller **Markov process**; under  $P^x$  started in  $x \in S$

## Running examples:

- E1  $S = (0, \infty)$ ,  $(X_t)$  Brownian motion killed when hitting the boundary
- E2  $S = \mathbb{R}^d$ ,  $(X_t)$  Brownian motion
- E3  $S = \mathbb{R}$ ,  $(X_t)$  subordinator

## Aim: Robust theory for entrance from boundary points; possibly

- ▶ entrance in infinite time
- ▶ entrance with “infinite activity”

→ Martin entrance theory

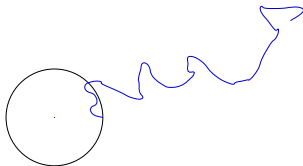
# Motivation

**Q:** What is a reasonable notion for a  $d$ -dimensional Brownian motion to enter from the boundary?

# Motivation

**Q:** What is a reasonable notion for a  $d$ -dimensional Brownian motion to enter from the boundary?

2d

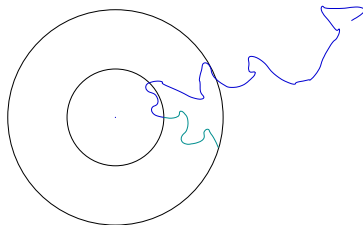




# Motivation

**Q:** What is a reasonable notion for a  $d$ -dimensional Brownian motion to enter from the boundary?

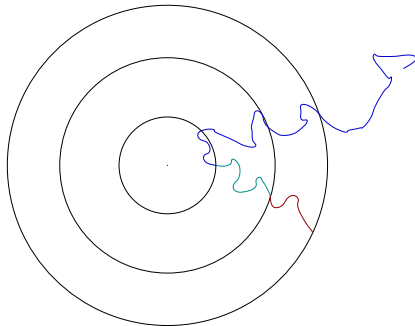
2d



# Motivation

**Q:** What is a reasonable notion for a  $d$ -dimensional Brownian motion to enter from the boundary?

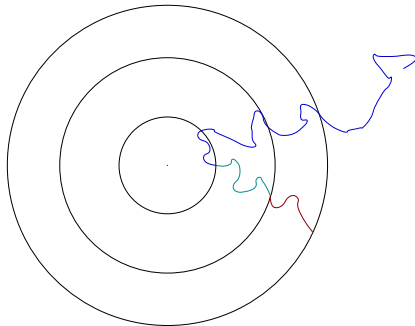
2d



# Motivation

**Q:** What is a reasonable notion for a  $d$ -dimensional Brownian motion to enter from the boundary?

2d



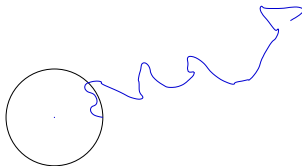
**Possibly:**

- ▶ infinite entrance (in particular, no canonical time parametrisation)

# Motivation

**Q:** What is a reasonable notion for a  $d$ -dimensional Brownian motion to enter from the boundary?

3d



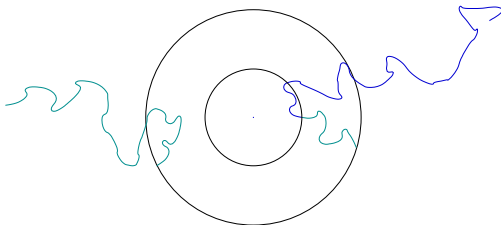
**Possibly:**

- ▶ infinite entrance (in particular, no canonical time parametrisation)

# Motivation

**Q:** What is a reasonable notion for a  $d$ -dimensional Brownian motion to enter from the boundary?

3d

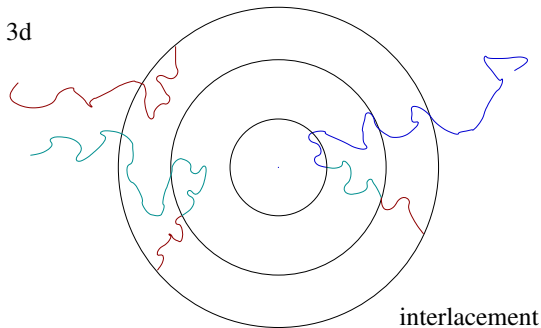


**Possibly:**

- ▶ infinite entrance (in particular, no canonical time parametrisation)

# Motivation

**Q:** What is a reasonable notion for a  $d$ -dimensional Brownian motion to enter from the boundary?



**Possibly:**

- ▶ infinite entrance (in particular, no canonical time parametrisation)
- ▶ infinitely many trajectories (random interlacement)

# Setting

Setting as in the analysis of random interacements!

- ▶  $D$ : set of càdlàg paths  $w : \mathbb{R} \rightarrow \bar{S}$  with **continuous** entrance from  $\partial$
- ▶  $D_+$ : set of càdlàg paths  $w : [0, \infty) \rightarrow \bar{S}$

# Setting

Setting as in the analysis of random interlacements!

- ▶  $D$ : set of càdlàg paths  $w : \mathbb{R} \rightarrow \bar{S}$  with **continuous** entrance from  $\partial$
- ▶  $D_+$ : set of càdlàg paths  $w : [0, \infty) \rightarrow \bar{S}$
- ▶  $D^*$ : paths in  $D$  modulo  $\sim$ , where  $(w_t) \sim (v_t)$  iff  $\exists s \in \mathbb{R}$

$$w_{s+t} = v_t, \quad \text{for all } t \in \mathbb{R}.$$



# Setting

Setting as in the analysis of random interacements!

- ▶  $D$ : set of càdlàg paths  $w : \mathbb{R} \rightarrow \bar{S}$  with **continuous** entrance from  $\partial$
- ▶  $D_+$ : set of càdlàg paths  $w : [0, \infty) \rightarrow \bar{S}$
- ▶  $D^*$ : paths in  $D$  modulo  $\sim$ , where  $(w_t) \sim (v_t)$  iff  $\exists s \in \mathbb{R}$

$$w_{s+t} = v_t, \quad \text{for all } t \in \mathbb{R}.$$

- ▶  $D_B, D_{B,+}, D_B^*$ : paths in  $D, D_+$  or  $D^*$  that hit the compact set  $B \subset S$

For  $w \in D_B$  or  $D_{B,+}$  set

$$T_B(w) = \inf\{t : w_t \in B\}, \quad H_B(w) = w_{T_B} \text{ and } \pi_B(w) = (w_{T_B+t})_{t \geq 0}$$

Note that  $H_B$  and  $\pi_B$  are also well-defined as maps on  $D_B^*$ !

# Entrance measure

**Definition:** A measure  $\xi$  on  $D^*$  is called **entrance measure**, if for any compact set  $B \subset S$

$$(F) \quad \xi(D_{B^*}) < \infty$$

$$(MP) \quad \xi \circ \pi_B^{-1} = P^{\xi \circ H_B^{-1}} \text{ or, more precisely, } \xi|_{D_B^*} \circ \pi_B^{-1} = P^{\xi|_{D_B^*} \circ H_B^{-1}}$$

# Entrance measure

**Definition:** A measure  $\xi$  on  $D^*$  is called **entrance measure**, if for any compact set  $B \subset S$

$$(F) \quad \xi(D_{B^*}) < \infty$$

$$(MP) \quad \xi \circ \pi_B^{-1} = P^{\xi \circ H_B^{-1}} \text{ or, more precisely, } \xi|_{D_B^*} \circ \pi_B^{-1} = P^{\xi|_{D_B^*} \circ H_B^{-1}}$$

**Remark:** To understand the **entrance measure** as a probabilistic object, one should rather consider the **Poisson point process**  $\Xi$  on  $D^*$  with intensity  $\xi$ .

# Entrance measure

**Definition:** A measure  $\xi$  on  $D^*$  is called **entrance measure**, if for any compact set  $B \subset S$

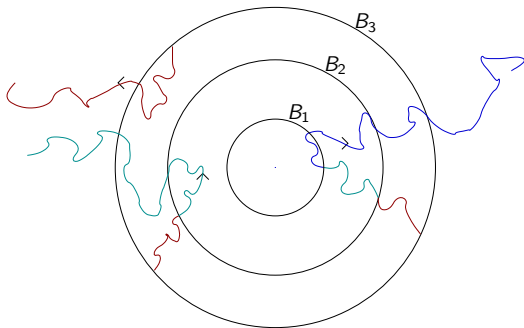
$$(F) \quad \xi(D_{B^*}) < \infty$$

$$(MP) \quad \xi \circ \pi_B^{-1} = P^{\xi \circ H_B^{-1}} \text{ or, more precisely, } \xi|_{D_B^*} \circ \pi_B^{-1} = P^{\xi|_{D_B^*} \circ H_B^{-1}}$$

**Remark:** To understand the **entrance measure** as a probabilistic object, one should rather consider the **Poisson point process**  $\Xi$  on  $D^*$  with intensity  $\xi$ .

- ▶ For  $d$ -dimensional Brownian motion ( $d \geq 3$ ) the corresponding point process  $\Xi$  is a random **interlacement** in the sense of Sznitman.
- ▶ (F) means that the number of path of  $\Xi$  that enter a compact set is almost surely finite.
- ▶ The Markov property (MP) means that  $\Xi|_{D_B^*}$  is constituted by a Poisson number of paths, that behave as the Feller process from their first entrance in  $B$ .
- ▶ If  $\xi$  has finite measure, then its normed version is a canonical choice for the distribution of a Markov process started at the boundary.

# Random interlacements/nested entrance regions



**Nested entrance regions:** Let  $(B_n)_{n \in \mathbb{N}}$  be a sequence of **compact** subsets of  $S$  with

$$B_n \subset \overset{\circ}{B}_{n+1} \text{ and } \bigcup B_n = S$$

**Assumptions:**

- ▶  $\forall x \in S : P^x(T_{B_1} < \infty) > 0$
- ▶  $\forall n \in \mathbb{N} : \lim_{m \rightarrow \infty} \limsup_{x \rightarrow \partial} P^x(T_{B_m} = T_{B_n} | T_{B_n} < \infty) = 0$

# Consistent entrance families

**Consistent entrance family:** Sequence  $(\mu_n)$  of distributions on  $(B_n)$  such that, for all  $n \in \mathbb{N}$ ,

$$P^{\mu_{n+1}}(X_{T_{B_n}} \in \cdot \mid T_{B_n} < \infty) = \mu_n(\cdot)$$

and

$$\lim_{k \rightarrow \infty} \mu_k(B_n) = 0.$$

# Consistent entrance families

**Consistent entrance family:** Sequence  $(\mu_n)$  of distributions on  $(B_n)$  such that, for all  $n \in \mathbb{N}$ ,

$$P^{\mu_{n+1}}(X_{T_{B_n}} \in \cdot \mid T_{B_n} < \infty) = \mu_n(\cdot)$$

and

$$\lim_{k \rightarrow \infty} \mu_k(B_n) = 0.$$

**Theorem:** There is a **one-to-one** correspondence between all entrance measures  $\xi$  with  $\xi(D_{B_1}^*) = 1$  and the set of consistent entrance families  $(\mu_n)$  for  $(B_n)_{n \in \mathbb{N}}$ . It is given by

$$\mu_n(\cdot) = \xi(H_{B_n} \in \cdot \mid D_{B_n}^*), \quad \text{for } n \in \mathbb{N}. \quad (1)$$

# Embedding into a compact space (Martin boundary)

**Aim:** Compactification of  $S$  such that new points are entrance measures.

**Attention:** The following is only true for a regularised notion of entrance region! For sake of simplicity we will ignore this and continue working with  $(B_n)$ .



# Embedding into a compact space (Martin boundary)

**Aim:** Compactification of  $S$  such that new points are entrance measures.

**Attention:** The following is only true for a regularised notion of entrance region! For sake of simplicity we will ignore this and continue working with  $(B_n)$ .

We associate  $x \in S$  with a sequence of distributions  $\mu^{(x)} = (\mu_1^{(x)}, \mu_2^{(x)}, \dots)$  via

$$\mu_n^{(x)}(\cdot) := P^x(H_{B_n} \in \cdot | T_{B_n} < \infty) \in \mathcal{M}_1(B_n).$$

# Embedding into a compact space (Martin boundary)

**Aim:** Compactification of  $S$  such that new points are entrance measures.

**Attention:** The following is only true for a regularised notion of entrance region! For sake of simplicity we will ignore this and continue working with  $(B_n)$ .

We associate  $x \in S$  with a sequence of distributions  $\mu^{(x)} = (\mu_1^{(x)}, \mu_2^{(x)}, \dots)$  via

$$\mu_n^{(x)}(\cdot) := P^x(H_{B_n} \in \cdot | T_{B_n} < \infty) \in \mathcal{M}_1(B_n).$$

**Theorem:** The embedding

$$\varphi : S \ni x \mapsto \mu^{(x)} \in \varphi(S) \subset \mathcal{M}_1(B_1) \times \mathcal{M}_1(B_2) \times \dots$$

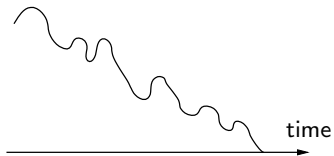
is a homeomorphism. There is a unique compactification  $\bar{S}$  of  $S$  for which  $\varphi$  extends to a homeomorphism  $\bar{\varphi} : \bar{S} \rightarrow \overline{\varphi(S)}$ .

For each  $x \in \Gamma := \bar{S} \setminus S$  the sequence  $\mu^{(x)}$  is a consistent entrance family and we associate  $x \in \Gamma$  to the corresponding  $D_{B_1}^*$ -normed entrance measure  $\xi^{(x)}$ .

# Examples

E1:  $S = (0, \infty)$ ,  $(X_t)$  Brownian motion  $\rightarrow \mathcal{S} = [0, \infty]$

$\xi^{(\infty)}$



backwards in time  $\text{Bes}(3)$   
finite entrance measure

$\xi^{(0)}$

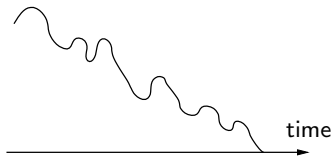


excursion measure of  $(|B_t|)$   
infinite entrance measure

# Examples

E1:  $S = (0, \infty)$ ,  $(X_t)$  Brownian motion  $\rightarrow \mathcal{S} = [0, \infty]$

$\xi^{(\infty)}$



backwards in time  $\text{Bes}(3)$   
finite entrance measure

$\xi^{(0)}$



excursion measure of  $(|B_t|)$   
infinite entrance measure

E2:  $S = \mathbb{R}^d$ ,  $(X_t)$  Brownian motion  $\rightarrow$

- ▶  $d \geq 2$ :  $S$  one-point compactification; **one** extremal entrance measure
- ▶  $d = 1$ :  $S = [-\infty, \infty]$ ; **two** extremal entrance measures

# Examples

E3:  $S = (-\infty, 0]$ ,  $(X_t)$  nonlattice subordinator  $\rightarrow \mathcal{S} = [-\infty, 0]$

**Note:**

- ▶  $\xi^{(-\infty)}(D^*) < \infty \Leftrightarrow$  overshoot distributions tight.

# Choquet type integral representation

**Note:** The family of entrance measure is a convex cone!

→ Choquet-type representations ?

**Theorem:** Let  $\xi$  be an entrance measure For  $\xi$ -almost all  $[w]$  the limit

$$\text{enter}([w]) := \lim_{t \downarrow T_\partial} w_t \in \Gamma$$

exists in  $\mathcal{S}$ , where  $T_\partial(w) = \inf\{t \in \mathbb{R} : w_t \in \mathcal{S}\}$  (entrance time).

# Choquet type integral representation

**Note:** The family of entrance measure is a convex cone!

→ Choquet-type representations ?

**Theorem:** Let  $\xi$  be an entrance measure For  $\xi$ -almost all  $[w]$  the limit

$$\text{enter}([w]) := \lim_{t \downarrow T_\partial} w_t \in \Gamma$$

exists in  $\mathcal{S}$ , where  $T_\partial(w) = \inf\{t \in \mathbb{R} : w_t \in \mathcal{S}\}$  (entrance time).

The point  $x \in \Gamma$  is called **extremal** (in  $\Gamma^{\text{ext}}$ ), iff

$$\text{enter}(w^*) = x, \text{ for } \xi^{(x)}\text{-almost all } w^*$$

**Theorem:** There is a **one-to-one** correspondence between

- ▶ the finite measures  $\nu$  on  $\Gamma^{\text{ext}}$  and
- ▶ the entrance measures  $\xi$ .

It is given by

$$\xi(\cdot) = \int \xi^{(x)}(\cdot) d\nu(x) \quad \text{and} \quad \nu(\cdot) = \int_{D_{B_1}^*} \mathbf{1}_{\{\text{enter}(w^*) \in \cdot\}} d\xi(w^*)$$

# Kolmogorov 0-1 law

**Theorem:** The entrance measure  $\xi$  satisfies a Kolmogorov 0-1 law for the entrance iff it is **extremal**.



# Kolmogorov 0-1 law

**Theorem:** The entrance measure  $\xi$  satisfies a Kolmogorov 0-1 law for the entrance iff it is **extremal**.

One may classify extremal entrance measures as follows:

	$\xi$ finite measure	$\xi$ infinite measure
entr. in finite time	$\xi^{(0)}$ for pssMp (entrance)	$\xi^{(0)}$ for pssMp (regular) $\xi^{(0)}$ for BM on $(0, \infty)$
in infinite time	$\xi^{(\infty)}$ for BM on $(0, \infty)$	$\xi^{(-\infty)}$ for subordinator with non-tight overshoots (non-lattice)

# Kolmogorov 0-1 law

**Theorem:** The entrance measure  $\xi$  satisfies a Kolmogorov 0-1 law for the entrance iff it is **extremal**.

One may classify extremal entrance measures as follows:

	$\xi$ finite measure	$\xi$ infinite measure
entr. in finite time	$\xi^{(0)}$ for pssMp (entrance)	$\xi^{(0)}$ for pssMp (regular) $\xi^{(0)}$ for BM on $(0, \infty)$
in infinite time	$\xi^{(\infty)}$ for BM on $(0, \infty)$	$\xi^{(-\infty)}$ for subordinator with non-tight overshoots (non-lattice)

Construction of a **Markov process** started at the boundary possible in two ways:

- ▶  $\xi$  finite measure with finite entrance  $\rightarrow$  take normed version of  $\xi$
- ▶  $\xi$  infinite measure with finite entrance  $\rightarrow$  **Itô's glueing construction** possible, iff

$$\int (1 - e^{-\text{lifetime}(w^*)}) d\xi(w^*) < \infty$$

# Invariant measures

**Q:** How to identify entrance measures?

Suppose that the Markov process is **strongly transient**:  $\forall x \in S, B \subset S$  comp.

$$E^x \left[ \int_0^\infty \mathbf{1}_B(X_t) dt \right] < \infty$$

# Invariant measures

**Q:** How to identify entrance measures?

Suppose that the Markov process is **strongly transient**:  $\forall x \in S, B \subset S$  comp.

$$E^x \left[ \int_0^\infty \mathbf{1}_B(X_t) dt \right] < \infty$$

**Theorem:** There is a **one-to-one** mapping between

- ▶ the **invariant measures**  $\nu$  (measures with  $\nu(B_1) < \infty$  and  $\nu = \nu P_t \forall t \geq 0$ )
- ▶ the **entrance measures**  $\xi$  with **infinite** time of entrance

It is given by

$$\nu(A) = \int_{D^*} \left( \int_{-\infty}^\infty \mathbf{1}_A(w_t) dt \right) d\xi([w]).$$

# Invariant measures

**Q:** How to identify entrance measures?

Suppose that the Markov process is **strongly transient**:  $\forall x \in S, B \subset S$  comp.

$$E^x \left[ \int_0^\infty \mathbf{1}_B(X_t) dt \right] < \infty$$

**Theorem:** There is a **one-to-one** mapping between

- ▶ the **invariant measures**  $\nu$  (measures with  $\nu(B_1) < \infty$  and  $\nu = \nu P_t \forall t \geq 0$ )
- ▶ the **entrance measures**  $\xi$  with **infinite** time of entrance

It is given by

$$\nu(A) = \int_{D^*} \left( \int_{-\infty}^\infty \mathbf{1}_A(w_t) dt \right) d\xi([w]).$$

**Application:** For (nonlattice) Lévy processes:

- ▶ all entrance measures have infinite time of entrance
- ▶ Choquet, Deny '60: the extremal invariant measures are of the form  $e^{-\alpha x} dx$  with  $\alpha \in \mathbb{R}$  being a solution to

$$\log E^0[e^{\alpha X_1}] = 0 \quad (0 \text{ trivial solution})$$

- ▶ correspondence between entrance measures of pssMp and Lévy process:  
→ Rivero-Fitzsimmons

# Idea of the proof

The following concepts are closely related:

1.  $\nu$  invariant measure
2. Measure  $Q$  on  $D$  such that

$$Q \circ (X_{t-t_0} : t \geq 0)^{-1} = P^\nu$$

for every  $t_0 \in \mathbb{R}$ .

3. Entrance measure  $\xi$  with property as in theorem.

# Idea of the proof

The following concepts are closely related:

1.  $\nu$  invariant measure
2. Measure  $Q$  on  $D$  such that

$$Q \circ (X_{t-t_0} : t \geq 0)^{-1} = P^\nu$$

for every  $t_0 \in \mathbb{R}$ .

3. Entrance measure  $\xi$  with property as in theorem.

1  $\Rightarrow$  2: Construction via Kolmogorov

# Idea of the proof

The following concepts are closely related:

1.  $\nu$  invariant measure
2. Measure  $Q$  on  $D$  such that

$$Q \circ (X_{t-t_0} : t \geq 0)^{-1} = P^\nu$$

for every  $t_0 \in \mathbb{R}$ .

3. Entrance measure  $\xi$  with property as in theorem.

1  $\Rightarrow$  2: Construction via Kolmogorov

2  $\Rightarrow$  3: Fix  $B \subset S$  compact. The measure  $Q|_{D_B} \circ (T_B, [X])^{-1}$  is invariant along shifts in first component. Factorisation theorem:  $\exists$  measure  $\xi_B$  on  $D_B^*$  with

$$Q|_{D_B} \circ (T_B, [X])^{-1} = \ell \otimes \xi_B$$

Take  $\xi = \lim_{B \uparrow S} \xi_B$ . (Similar as construction of Palm distributions for stationary point processes.)



## Concluding remarks

We developed the theory of Martin entrance boundaries along **hitting distributions** (balayage).

# Concluding remarks

We developed the theory of Martin entrance boundaries along **hitting distributions** (balayage).

We rely on techniques that are common in the field of probability:

- ▶ The correspondence **consistent entrance families**  $\leftrightarrow$  **entrance measure** and the **Choquet** type result parallels the construction of Gibbs measures in statistical mechanics.
- ▶ The proof of convergence in the definition of **enter** is based on martingale arguments (Doob's upcrossing inequality; similar as in classical Martin boundary theory)
- ▶ The correspondence **invariant measures**  $\leftrightarrow$  **entrance measures** parallels the construction of Palm distributions for stationary point processes.

# Concluding remarks

We developed the theory of Martin entrance boundaries along **hitting distributions** (balayage).

We rely on techniques that are common in the field of probability:

- ▶ The correspondence **consistent entrance families**  $\leftrightarrow$  **entrance measure** and the **Choquet** type result parallels the construction of Gibbs measures in statistical mechanics.
- ▶ The proof of convergence in the definition of **enter** is based on martingale arguments (Doob's upcrossing inequality; similar as in classical Martin boundary theory)
- ▶ The correspondence **invariant measures**  $\leftrightarrow$  **entrance measures** parallels the construction of Palm distributions for stationary point processes.

The talk is based on work in progress and we plan to finish the preprint in two months time.