On entrance boundaries and a generalised notion of an interlacement

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Motivation

Central Question: How to start a Markov process from a boundary point?

Lévy processes/pssMps: Bertoin, Caballero, Chaumont, Döring, Fitzsimmons, Kyprianou, Lamperti, Pardo, Rivero, Savov, Yor, ..

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Related to: Potential analysis, Martin boundaries, ...

General theory: Doob, Dynkin, Feller, Hunt, Kunita, Martin, Pinsky, Watanabe, ..

Martin kernels, etc: Bogdan, Byczkowski, Damek, Vondracek, ..

The Markov process

Setting:

- S locally compact Polish space (non-compact), the state space
- $\bar{S} = S \cup \{\partial\}$ one-point compactification
- $(X_t)_{t\geq 0}$ càdlàg Feller Markov process; under P^x started in $x \in S$

Running examples:

E1 $S = (0, \infty)$, (X_t) Brownian motion killed when hitting the boundary

E2 $S = \mathbb{R}^d$, (X_t) Brownian motion

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Aim: Robust theory for entrance from boundary points; possibly

- entrance in infinite time
- entrance with "infinite activity"

$\to$ Martin entrance theory

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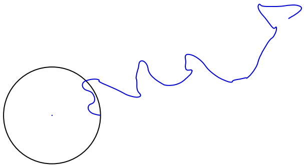

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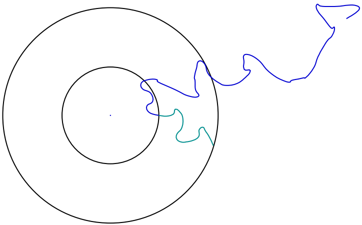

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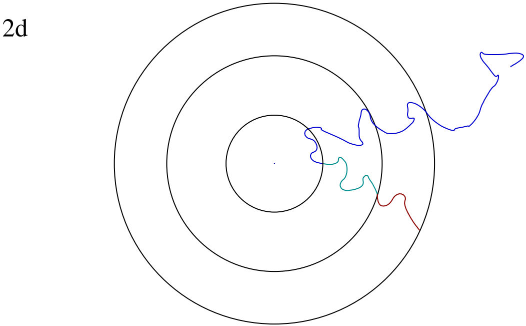

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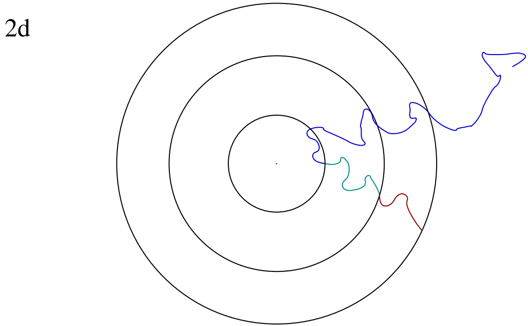

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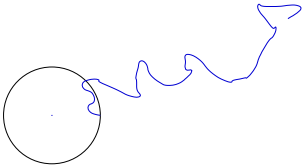

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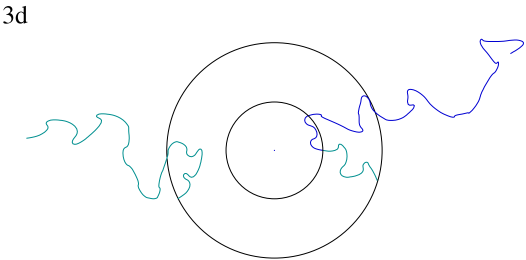

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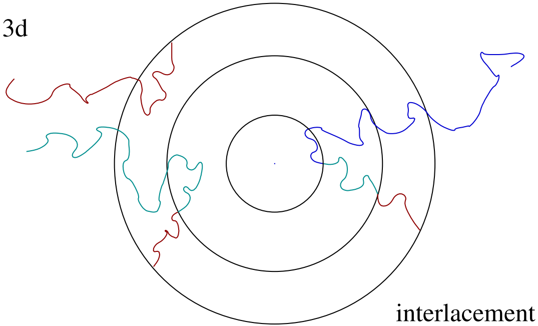

Possibly:

- infinite entrance (in particular, no canonical time parametrisation)
- infinitely many trajectories (random interlacement)

Setting

Setting as in the analysis of random interlacements!

- D: set of càdlàg paths $w : \mathbb{R} \to \bar{S}$ with continuous entrance from ∂
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- $D_B, D_{B,+}, D_B^*$: paths in D, D_+ or D^* that hit the compact set $B \subset S$

For $w \in D_B$ or D_{B_+} set

$$T_B(w) = \inf\{t : w_t \in B\}, \; H_B(w) = w_{T_B} \text{ and } \pi_B(w) = (w_{T_B+t})_{t \geq 0}$$

Note that H_B and π_B are also well-defined as maps on D_B^*!

Entrance measure

Definition: A measure ξ on D^* is called entrance measure, if for any compact set $B \subset S$

(F) $\xi(D_{B^*}) < \infty$

(MP) $\xi \circ \pi_B^{-1} = P^{\xi \circ H_B^{-1}}$ or, more precisely, $\xi|_{D_B^*} \circ \pi_B^{-1} = P^{\xi|_{D_B^*} \circ H_B^{-1}}$

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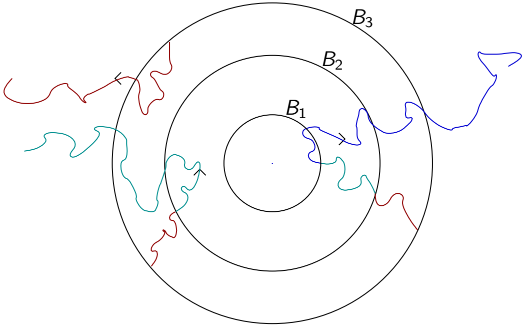
- For d-dimensional Brownian motion ($d \geq 3$) the corresponding point process Ξ is a random interlacement in the sense of Sznitman.
- (F) means that the number of path of Ξ that enter a compact set is almost surely finite.
- The Markov property (MP) means that $\Xi|_{D_B^*}$ is constituted by a Poisson number of paths, that behave as the Feller process from their first entrance in B.
- If ξ has finite measure, then its normed version is a canonical choice for the distribution of a Markov process started at the boundary.

Random interlacements/nested entrance regions



Nested entrance regions: Let $(B_n)_{n\in\mathbb{N}}$ be a sequence of compact subsets of S with

$$B_n \subset \dot{B}_{n+1} \quad \text{and} \quad \bigcup B_n = S$$

Assumptions:

- $\forall x \in S : \; P^x(T_{B_1} < \infty) > 0$
- $\forall n \in \mathbb{N} : \; \lim_{m\to\infty} \limsup_{x\to\partial} P^x(T_{B_m} = T_{B_n} | T_{B_n} < \infty) = 0$

Consistent entrance families

Consistent entrance family: Sequence (μ_n) of distributions on (B_n) such that, for all $n \in \mathbb{N}$,

$$P^{\mu_{n+1}}(X_{T_{B_n}} \in \cdot \mid T_{B_n} < \infty) = \mu_n(\cdot)$$

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Theorem: There is a one-to-one correspondence between all entrance measures ξ with $\xi(D^*_{B_1}) = 1$ and the set of consistent entrance families (μ_n) for $(B_n)_{n\in\mathbb{N}}$. It is given by

$$\mu_n(\cdot) = \xi\big(H_{B_n} \in \cdot \,\big|\, D^*_{B_n}\big), \qquad \text{for } n \in \mathbb{N}. \tag{1}$$

Embedding into a compact space (Martin boundary)

Aim: Compactification of S such that new points are entrance measures.

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Theorem: The embedding

$$\varphi : S \ni x \mapsto \mu^{(x)} \in \varphi(S) \subset \mathcal{M}_1(B_1) \times \mathcal{M}_1(B_2) \times \dots$$

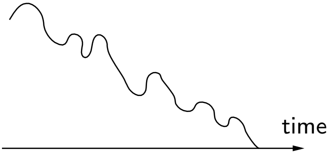
is a homeomorphism. There is a unique compactification $\mathcal{S}$ of S for which φ extends to a homeomorphism $\bar{\varphi} : \mathcal{S} \to \overline{\varphi(S)}$.

For each $x \in \Gamma := \mathcal{S} \backslash S$ the sequence $\mu^{(x)}$ is a consistent entrance family and we associate $x \in \Gamma$ to the corresponding $D^*_{B_1}$-normed entrance measure $\xi^{(x)}$.

Examples

E1: $S = (0, \infty)$, (X_t) Brownian motion $\rightarrow$ $\mathcal{S} = [0, \infty]$

$\xi^{(\infty)}$



backwards in time $\mathrm{Bes}(3)$
finite entrance measure

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excursion measure of $(|B_t|)$
infinite entrance measure

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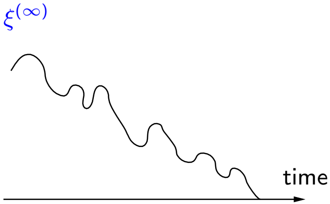

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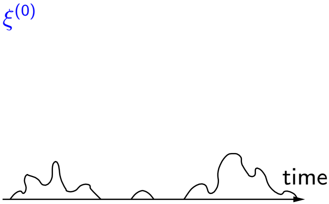

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E2: $S = \mathbb{R}^d$, (X_t) Brownian motion $\to$

- $d \geq 2$: $\mathcal{S}$ one-point compactification; one extremal entrance measure
- $d = 1$: $\mathcal{S} = [-\infty, \infty]$; two extremal entrance measures

Examples

E3: $S = (-\infty, 0]$, (X_t) nonlattice subordinator $\rightarrow \mathcal{S} = [-\infty, 0]$

Note:

- $\xi^{(-\infty)}(D^*) < \infty \;\Leftrightarrow\;$ overshoot distributions tight.

Choquet type integral representation

Note: The family of entrance measure is a convex cone!
$\rightarrow$ Choquet-type representations ?

Theorem: Let ξ be an entrance measure For ξ-almost all $[w]$ the limit

$$\mathrm{enter}([w]) := \lim_{t \downarrow T_\partial} w_t \in \Gamma$$

exists in $\mathcal{S}$, where $T_\partial(w) = \inf\{t \in \mathbb{R} : w_t \in S\}$ (entrance time).

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The point $x \in \Gamma$ is called extremal (in Γ^{ext}), iff

$$\mathrm{enter}(w^*) = x, \quad \text{for } \xi^{(x)}\text{-almost all } w^*$$

Theorem: There is a one-to-one correspondence between

- the finite measures ν on Γ^{ext} and
- the entrance measures ξ.

It is given by

$$\xi(\cdot) = \int \xi^{(x)}(\cdot)\, \mathrm{d}\nu(x) \quad \text{and} \quad \nu(\cdot) = \int_{D^*_{B_1}} \mathbb{1}_{\{\mathrm{enter(w^*)} \in \cdot\}}\, \mathrm{d}\xi(w^*)$$

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Construction of a Markov process started at the boundary possible in two ways:

- ξ finite measure with finite entrance $\to$ take normed version of ξ
- ξ infinite measure with finite entrance $\to$ Itô's glueing construction possible, iff

$$\int (1 - e^{-\mathrm{lifetime}(w^*)})\, d\xi(w^*) < \infty$$

Invariant measures

Q: How to identify entrance measures?

Suppose that the Markov process is strongly transient: $\forall x \in S, B \subset S$ comp.

$$E^x\left[\int_0^\infty \mathbb{1}_B(X_t)\,\mathrm{d}t\right] < \infty$$

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- the invariant measures ν (measures with $\nu(B_1) < \infty$ and $\nu = \nu P_t \ \forall t \geq 0$)
- the entrance measures ξ with infinite time of entrance

It is given by

$$\nu(A) = \int_{D^*}\left(\int_{-\infty}^{\infty} \mathbb{1}_A(w_t)\,\mathrm{d}t\right)\mathrm{d}\xi([w]).$$

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Application: For (nonlattice) Lévy processes:

- all entrance measures have infinite time of entrance
- Choquet, Deny '60: the extremal invariant measures are of the form $e^{-\alpha x}\,\mathrm{d}x$ with $\alpha \in \mathbb{R}$ being a solution to

$$\log E^0[e^{\alpha X_1}] = 0 \qquad (0 \ \text{ trivial solution})$$

- correspondence between entrance measures of pssMp and Lévy process: $\to$ Rivero-Fitzsimmons

Idea of the proof

The following concepts are closely related:

1. ν invariant measure
2. Measure Q on D such that

$$Q \circ (X_{t-t_0} : t \geq 0)^{-1} = P^{\nu}$$

   for every $t_0 \in \mathbb{R}$.

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1 $\Rightarrow$ 2**:** Construction via Kolmogorov

2 $\Rightarrow$ 3**:** Fix $B \subset S$ compact. The measure $Q|_{D_B} \circ (T_B, [X])^{-1}$ is invariant along shifts in first component. Factorisation theorem: $\exists$ measure ξ_B on D_B^* with

$$Q|_{D_B} \circ (T_B, [X])^{-1} = \ell \otimes \xi_B$$

Take $\xi = \lim_{B \uparrow S} \xi_B$. (Similar as construction of Palm distributions for stationary point processes.)

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The talk is based on work in progress and we plan to finish the preprint in two months time.