Free infinitely divisible distributions from a point of view of subordinators and mixtures

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Motivation

- $B_t(\omega)$: BM
- $\mathrm{B}_t(\omega) = (B_{ij}(\omega))_{i,j=1}^d$: $d \times d$ Hermitian (symmetric) matrices valued BM
- $(\lambda_t^{(i)})_{i=1}^d$: Dyson BM (eigenvalue processes)
  i.e. $\mathrm{B}_t(\omega) = U(\omega)\mathrm{diag}(\lambda_t^{(1)}(\omega), \ldots, \lambda_t^{(d)}(\omega))U^*(\omega)$
- Measure valued process:
  $\mu_t^{(d)}(B, \omega) := \sum_{i=1}^d \delta_{\lambda_t^{(i)}}(B, \omega)$ for $B \in \mathcal{B}(\mathbb{R})$
- Laws of Free Brownian motion:
  $\lim_{d\to\infty} \mu_t^{(d)}(B) = \mathrm{w}_t$ a.s.

Motivation

- **Purpose: we investigate matrix-valued Lévy processes.**
- **We try to understand spectral distri. of large size Lévy RM.**
- **We want to treat mixture and subordinator for Lévy RM.**
- **To investigate spectral distri. of product of RM, we use "free probablity".**

Motivation

- What's free probability??
- Voiculescu $(80's)$

$$\text{Free Probability} \\ = \text{Non-commutative Probability} + \text{Free Independence}$$

- Why do we consider free probability??
  Voiculescu (1991)

Application to Random Matrices

- Free probability can be regarded as tool to deal with eigenvalue distribution of sum and product of random matrices.
- Researchers are interested in free probability itself.
  Operator algebra, algebraic geometry,...

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§1
Short Introduction of Free Probability

Probability Space, Distributions, Moments

Concepts in probability theory

- **Probability space:** $(\Omega, \mathcal{F}, \mathbb{P})$
- **Random variable:** $X(\omega)$
- **Distribution of $X(\omega)$:** $\mu_X \cdots$
- **Moment sequence:** $X \sim \mu_X \in \mathcal{P}$

$$
\begin{aligned}
\mathbb{E}[X^0] &= \int_{\mathbb{R}} 1\, \mu_X(dx) = 1, \\
\mathbb{E}[X] &= \int_{\mathbb{R}} x\, \mu_X(dx) \\
\mathbb{E}[X^2] &= \int_{\mathbb{R}} x^2 \mu_X(dx) \\
&\ldots
\end{aligned}
$$

- **Moment problem: moments determine its distribution uniquely if support of distribution is bounded.**

- Independence(Calculation Rule)
  Random variables X and Y are independent

$$\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y]$$

- Central Limit Theorem
  $\{X_n\}_{n=1}^{\infty}$: i.i.d., mean 0, variance1.

$$\text{Law of } \frac{X_1 + \cdots + X_n}{\sqrt{n}} \xrightarrow{d} N(0,1) \text{ as } n \to \infty$$

- We calculate all moments of this scale and take its limit.
  We obtain moments of standard Gaussian r.v.

Non-commutative Probablity Space (NCPS)

- $\mathcal{A}$: $*$-algebra with $\mathbb{1} \in \mathcal{A}$.
- τ : state on $\mathcal{A}$ i.e. linear functional on $\mathcal{A}$, $\tau(XX^*) \geq 0, \tau(X^*) = \overline{\tau(X)}$, $\tau(1) = 1 \ldots$.
- $(\mathcal{A}, \tau)$ is called NCPS.
- $X \in \mathcal{A}$: (non-commutative) random variable
- τ : a map from a random variable to its mean.
- Moment sequence: $X \sim \mu_X$

$$\tau(X^0) = \int_{\mathbb{R}} 1\, \mu_X(dx) = 1,$$
$$\tau(X) = \int_{\mathbb{R}} x\, \mu_X(dx),$$
$$\tau(X^2) = \int_{\mathbb{R}} x^2 \mu_X(dx), \ldots$$

Ex.

$$\mathcal{A} := \mathbb{M}_d(L^\infty(\Omega, \mathcal{F}, P))$$
$$\tau_d(X) := \frac{1}{d}\mathbb{E}[\mathrm{Tr}(X)], \quad X \in \mathcal{A}$$

In particular, X is selfadjoint $(X = X^*)$
$\Rightarrow$ Spectral distribution μ_X^d of X

$$\mu_X^d(B) = \frac{1}{d}\sum_{i=1}^{d} \delta_{\lambda_i(\omega)}(B),$$

where $\lambda_1(\omega) \leq \cdots \leq \lambda_d(\omega)$: (random) eigenvalues of X.

Spectral distribution of Large Hermite Gaussian

$X_d = [X_{i,j}]_{1 \le i,j \le d}$: $d \times d$Gaussian Unitary Ensemble

- $X_{i,j} = \overline{X_{j,i}}$
- $X_{i,i} \sim N(0, 1/d)(i = j)$
- $\Re(X_{i,j}), \Im(X_{i,j}) \sim N(0, 1/2d)(i \ne j)$
- $\{X_{i,i}, \Re(X_{i,j}), \Im(X_{i,j}) : 1 \le i < j \le d\}$ are independent.

Wigner

$$\tau_d(X_d^k) \to \frac{1}{k+1}\binom{2k}{k} =: C_k, \quad (d \to \infty)$$

C_k: Catalan number, moment sq. of the semi-circle distribution

Free Gaussian distribution: semi-circle distribution

- $\mathbf{w}_{m,\sigma^2}$: semi-circle distribution with mean $m \in \mathbb{R}$, variance $\sigma^2 > 0$
$\overset{\text{def}}{\Longleftrightarrow}$
$$\mathbf{w}_{m,\sigma^2}(dx) = \frac{1}{2\pi\sigma^2}\sqrt{4\sigma^2 - (x-m)^2}\,\mathbb{1}_{[-2\sigma+m, 2\sigma+m]}(x)dx.$$
- In particular, the standard semi-circle distribution (mean 0, variance 1) is denoted by $\mathbf{w}$.

Free independence (1)

Def.

Let $(\mathcal{A}, \tau)$ be NCPS.

$(X_i)_{i \in I}$ is free ind. on $(\mathcal{A}, \tau)$ $\overset{\text{def}}{\Longleftrightarrow} \forall n \in \mathbb{N}$,

- Let $X_k^\circ := X_k - \tau(X_k) = 0$,
- whenever $i(1) \neq i(2) \neq \cdots \neq i(n)$,
  $\Rightarrow \tau(X_{i(1)}^\circ X_{i(2)}^\circ \cdots X_{i(n)}^\circ) = 0$

Rem

Assume that $\mathcal{A}_1$ and $\mathcal{A}_2$ are free ind., $X_1 \in \mathcal{A}_1$, $\tau(X_1) = 0$, $X_2 \in \mathcal{A}_2$ and $\tau(X_2) = 0$. $\Rightarrow \tau(X_1 X_2 X_1 X_2) = 0$. If we also assume that random variables are commutative w.r.t. product, then distribution of X_1 and X_2 is δ_0.

Def. (Free convolutions)

X_1, X_2: f. ind. and their distributions are μ_1, μ_2.
$\Rightarrow$ law of $X_1 + X_2$: $\mu_1 \boxplus \mu_2$. $\boxplus$: (additive) free convolution.
law of $\sqrt{X_1} X_2 \sqrt{X_1}$: $\mu_1 \boxtimes \mu_2$. $\boxtimes$: multiplicative free convolution.

Realization of free convolution via RM

Thm. (Voiculescu+so many people)

- $\{U_d\}$: Haar Unitary RM
- $\{A_d\}$, $\{B_d\}$: Hermite RM with spectral distributions

$$\frac{1}{d}\sum_{i=1}^{d}\delta_{\lambda_i^{A_d}(\omega)} \overset{dist.}{\longrightarrow} \mu_1, \quad \frac{1}{d}\sum_{i=1}^{d}\delta_{\lambda_i^{B_d}(\omega)} \overset{dist.}{\longrightarrow} \mu_2 \text{ (a.s.)},$$

as $d \to \infty$. $\{U_d\}$, $\{A_d\}$, $\{B_d\}$ are classical ind.

$$\frac{1}{d}\sum_{i=1}^{d}\delta_{\lambda_i^{A_d+U_dB_dU_d^*}(\omega)} \overset{dist.}{\longrightarrow} \mu_1 \boxplus \mu_2. \quad \text{(a.s.)}$$

as $d \to \infty$.

Why do we consider free probability?

We want to deal with multi. RM.

$$\begin{cases} \{X_d^{(n)} : d \in \mathbb{N}\}_{n=1}^{k}\text{: classical ind.} d \times d \text{Hermite GRM} \\ \{X^n\}\text{: free iid r.v. sq., with the semi-circle distribution} \end{cases}$$

$\Rightarrow \forall P(X_1, \ldots, X_k) \in \mathbb{C}\langle X_1, \ldots, X_k \rangle$: non-commutative polynomial

$$\underbrace{\tau_d(P(X_d^{(1)}, \ldots, X_d^{(k)}))}_{\text{mixed moments of RM}} \xrightarrow{d \to \infty} \underbrace{\tau(P(X^{(1)}, \ldots, X^{(k)}))}_{\text{mixed moments of free random variable}} .$$

Central Limit Theorem (CLT)

$\mathcal{L}(X)$: distribution of r.v. X.

Thm. (free CLT, Voiculescu)

$\{X_n\}$: free iid sq., mean m, variance σ^2.
As $n \to \infty$,

$$\mathcal{L}\left(\frac{X_1 + \cdots + X_n - nm}{\sqrt{n\sigma^2}}\right) \xrightarrow{dist.} \mathbf{w}.$$

Law of Lareg Number(LLN) and Poisson type limit theorem are proved.

§2 Infinitely Divisible Distributions in Free Probability

Infinitely Divisible Distribution (inf. div. distr.)

$\mathcal{P}$: the set of all Borel prob. meas. on $\mathbb{R}$.

$\mathcal{P}_+$: the set of all Borel prob. meas. on $[0, \infty)$.

$\mathcal{P}_s$: the set of all symmetric[1] Borel prob. meas. on $\mathbb{R}$.

Def.

μ: free inf. div. distr. $\overset{\text{def}}{\Longleftrightarrow}$ $\forall n \in \mathbb{N}$, $\exists \mu_n \in \mathcal{P}$ s.t.

$$\mu = \underbrace{\mu_n \boxplus \mu_n \boxplus \cdots \boxplus \mu_n}_{n \text{ times}}$$

$$(\mu = \underbrace{\mu_n * \mu_n * \cdots * \mu_n}_{n \text{ times}})$$

$I^{\boxplus}$: the set of all free inf. div. distr. on $\mathbb{R}$.

I^{*}: the set of all (classical) inf. div. distr. on $\mathbb{R}$..

[1] symmetric means $\mu(B) = \mu(-B)$, $-B = \{x \in \mathbb{R} : -x \in B\}$

Why do we consider inf. div. distri.?

Marginal distri. of Lévy processes $\leftrightarrow$inf. div. distri.

$\{X(t)\}_{t\geq 0}$: a Lévy process

- **independent incre.**
- **time homogenous**
- **$X(0) = 0$ a.s.**
- **its path is stoch. conti..**

Ex. of Lévy process: B.M., Poisson pr., ...

$$\text{Law of } X(1) = \mu \Rightarrow \mathcal{L}(X(t)) = \mu^{*t}$$

Anaylytic tools (1)

Cauchy transform : for $\mu \in \mathcal{P}$,

$$G_\mu(z) := \int_{\mathbb{R}} \frac{1}{z-x} d\mu(x), \quad z \in \mathbb{C} \backslash \mathbb{R}.$$

$F_\mu(z) := 1/G_\mu(z), \quad z \in \mathbb{C} \backslash \mathbb{R}.$

$\Gamma_{\alpha,\beta} := \{z \in \mathbb{C}; \Im(z) > \alpha|\Re(z)|, \Im(z) > \beta\}.$

$F_\mu^{-1}(z)$: right inverse of $F_\mu(z)$,

i.e. $F_\mu(F_\mu^{-1}(z)) = z, \quad z \in \Gamma_{\alpha,\beta}.$

Anaylytic tools (2)

Def. (Free cumulant transform)

$$R_\mu(z) = zF_\mu^{-1}(1/z) - 1, \quad 1/z \in \Gamma_{\alpha,\beta},$$

is called R-transform or free cumulant transform.

Prop. (Voiculescu)

$$R_{\mu \boxplus \nu}(z) = R_\mu(z) + R_\nu(z), \quad 1/z \in \Gamma_{\alpha,\beta}$$

free cumulant transform $R_\mu(z) \leftrightarrow$ cumulant transform[2] $\mathcal{C}_\mu(z)$

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[2] $\mathcal{C}_\mu(z) = \log\left(\int_{\mathbb{R}} e^{itx} d\mu(x)\right)$

Example of free convolution

Ex.

$\mathbf{d} = \frac{1}{2}\delta_{-1} + \frac{1}{2}\delta_1 \Rightarrow \mathbf{d} \boxplus \mathbf{d}$: symm. arcsine distri.: density

$$f(x) = \frac{1}{\pi\sqrt{4-x^2}} \mathbf{1}_{[-2,2]}(x)$$

Lévy–Khintchine representation

Prop. (Lévy–Khintchine representation)

$\mu \in I^*$
$\overset{\text{def}}{\Longleftrightarrow} \exists 1 a \geq 0$, $\exists 1 c \in \mathbb{R}$, $\exists 1 \nu$: Lévy measure [3] s.t.

$$\mathcal{C}_\mu(z) = -\frac{1}{2}az^2 + icz + \int_{\mathbb{R}} \left(e^{izx} - 1 - izx\mathbf{1}_{[-1,1]}(x) \right) \nu(dx).$$

Rem.

a is Gaussian part, c is shift part. If you give (a, ν, c), inf. div. distri. are uniquely determined. (a, ν, c) is called triplet of LK rep..

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[3] a measure satisfying $\int_{\mathbb{R}} (1 \wedge |x|^2) \nu(dx) < \infty$ and $\nu(\{0\}) = 0$

Prop. (Free Lévy–Khintchine representation)

$\mu \in I^{\boxplus}$

$\overset{\text{def}}{\Longleftrightarrow} \exists 1 a \geq 0$, $\exists 1 c \in \mathbb{R}$, $\exists 1 \nu$: Lévy measure [4] s.t.

$$R_\mu(z) = az^2 + cz + \int_{\mathbb{R}} \left(\frac{1}{1 - xz} - 1 - xz \mathbf{1}_{[-1,1]}(x) \right) \nu(dx).$$

Rem.

a is semi-circle part, c is shift part. If you give (a, ν, c), inf. div. distri. are uniquely determined. (a, ν, c) is called triplet of free LK rep..

[4] a measure satisfying $\int_{\mathbb{R}} (1 \wedge |x|^2) \nu(dx) < \infty$ and $\nu(\{0\}) = 0$

Bercovici–Pata Bijection

Def.

$\Lambda : I^* \to I^{\boxplus}$: Bercovici–Pata Bijection

$\stackrel{\text{def}}{\Longleftrightarrow} \mu \in I^*$ with $(a, \nu, c) \mapsto \Lambda(\mu) \in I^{\boxplus}$ with (a, ν, c).

- **μ: Gaussian distri. $\leftrightarrow$ $\Lambda(\mu)$: semi-circle distri. $\mathbf{w}$**
- **μ: Poisson distri. $\leftrightarrow$ $\Lambda(\mu)$: Marchenko–Pastur distri. $\mathbf{m}$**
- **μ: Cauchy distri. $\leftrightarrow$ $\Lambda(\mu)$: Cauchy distri.**
- **μ: positive 1/2stable distri. $\leftrightarrow$ $\Lambda(\mu)$: beta distri. of 2nd kind. (Perez-Abreu, S)**

Important Distributions

Marchenko–Pastur distri.(free Poisson distri.) :

$$\mathbf{m}_\lambda(dx) = \mathbf{m}_{\lambda,1}(dx) = \min\{(1-\lambda), 0\}\delta_0(dx) + \frac{1}{2\pi x}\sqrt{4\lambda - (x-(1+\lambda))^2}\mathbf{1}_{[(1-\sqrt{\lambda})^2,(1+\sqrt{\lambda})^2]}(x)dx$$

free one-side 1/2stable distri. $=$ beta distri. of 2nd kind;
$c > 0, b > 0.$

$$f(x) = \frac{c}{\pi}\frac{\sqrt{(x-b)-\frac{c^2}{4}}}{(x-b)^2} \qquad \left(x > \frac{c^2}{4} + b\right)$$

Free regular inf.div. distri.

$\sigma \in I^{\boxplus}$: **free regular inf.div. distri.** $\overset{\text{def}}{\Longleftrightarrow} \exists 1 b_\sigma \geq 0, \exists 1 \nu_\sigma$: Lévy measure s.t.

- $\nu_\sigma((-\infty, 0]) = 0$
- $\int_0^\infty (1 \wedge x) \nu_\sigma(\mathrm{d}x) < \infty$
- $$R_\sigma(z) = b_\sigma z + \int_{(0,\infty)} \left(\frac{1}{1 - zx} - 1 \right) \nu_\sigma(dx)$$

$I_{r+}^{\boxplus}$: the set of all free regular inf.div. distri..

$$I^{\boxplus} \cap \mathcal{P}_+ \neq I_{r+}^{\boxplus}$$

Ex. (Counter examples)

- shifted semicircle distri.
- the one-side **1/2**–free stable with small positive drift.

Def.

μ: compound free Poisson distri. $\overset{\text{def}}{\iff}$ $\exists \sigma \in \mathcal{P}$ with $\sigma(\{0\}) = 0$ $\exists c > 0$ s.t.

$$R_\mu(z) = c \int_{\mathbb{R}_+} \left(\frac{1}{1 - zx} - 1 \right) \sigma(dx), \quad z \in \mathbb{C}^-.$$

Prop.

(1) $\mu \in I_{r+}^{\boxplus} \cup I_s^{\boxplus}$:compound free Poisson distri. $\Rightarrow \exists \sigma \in \mathcal{P}$ with $\sigma(\{0\}) = 0$ $\exists c > 0$ s.t. $\mu^{\boxplus c} = \mathbf{m} \boxtimes \sigma$.

(2) $\sigma \in \mathcal{P}_+ \cup \mathcal{P}_s$. $\mu = \mathbf{m} \boxtimes \sigma \Rightarrow \mu$: compound free Poisson distri..

§3

Mixture and Infinite Divisibility in Free Probability

V. Perez-Abreu and N. Sakuma

Free infinite divisibility of free multiplicative mixtures of the Wigner distribution.

J. Theor. Probab., 25, 100–121, (2012).

Variance mixture, type$G$

$Z_{\sigma^2} \sim N(0,\sigma^2) \Rightarrow Z_{\sigma^2} \overset{d}{=} \sigma Z$.

Randomize the variance σ^2.

$Z \sim N(0,1)$.

X: **variance mixture** $\overset{\text{def}}{\Longleftrightarrow} \exists V > 0$ **s.t.** $Z \perp\!\!\!\perp V$, $X \overset{\mathbf{d}}{=} VZ$

$$\begin{cases} X\text{: variance mixture of normal distribution} \\ V^2\text{: inf. div. distributed} \end{cases}$$

$\Rightarrow X$ **is inf. div..**

Thm. (Kelker)

There exist the case that V^2 is not $*$-ID but VZ is $*$-ID.

From a view point of Lévy process

Subordination

$\{X(t)\}_{t\geq 0}$:Lévy process
$\{Z(t)\}_{t\geq 0}$: increasing Lévy process
$\{X(t)\}_{t\geq 0}$ and $\{Z(t)\}_{t\geq 0}$ are indep.

$$\Rightarrow \{X(Z(t))\}_{t\geq 0}\text{:Lévy process}$$

In particular if $X(t)$ is B.M. $B(t)$,

$$B(Z(t)) \stackrel{d}{=} \sqrt{Z(t)}B(1)$$

$B(1) \sim N(0,1)$, $\mathcal{L}(Z(t)) \in I^*$,
$B(1)$and $Z(t)$ are indep.
$\Rightarrow \sqrt{Z(t)}B(1) \in I^*$

Def.

- μ: Variance Mixture of semi-circle distribution
  $\overset{\text{def}}{\Longleftrightarrow} \exists\, \overline{\sigma} \in \mathcal{P}_+$ s.t. $\mu = \overline{\sigma} \boxtimes \mathbf{w}$.
- μ: Type W $\overset{\text{def}}{\Longleftrightarrow} \exists\, \overline{\sigma} \in \mathcal{P}_+$ s.t. $\mu = \overline{\sigma} \boxtimes \mathbf{w} \in I^{\boxplus}$.

Problem

This class is subclass of $\mathcal{P}_s$

- How large is this class?
- How do we characterize it?

Prop. (Symmetric but not VM of S-C distri.)

$\mathbf{a}$: symm. arcsine distri. on $(-1, 1)$.

$\Rightarrow \nexists \lambda \in \mathcal{P}_+$ s.t. $\mathbf{a} = \lambda \boxtimes \mathbf{w}$. $\Rightarrow \mathbf{a}$ is not VM of S-C distri..

Necessary and Sufficient condition for Free ID of VM of S-C.

Thm. (Pérez-Abreu and S)

(1) $\overline{\sigma} \in \mathcal{P}_+$.
$\mu = \overline{\sigma} \boxtimes \mathbf{w} \in I^{\boxplus} \Leftrightarrow \sigma = \overline{\sigma} \boxtimes \overline{\sigma} \in I^{\boxplus}_{r+}$.

(2) μ: VM of S-C distri. $\Rightarrow \mu^{(2)}$: compound free Poisson distri., where $\mu^{(2)}$ means that $\mu^{(2)} = \mathcal{L}(X^2)$ if $\mu = \mathcal{L}(X)$.

Rem

We obtain Necessary and Sufficient condition for Free ID of VM of S-C.

Examples for the condition

Examples for $\sigma = \overline{\sigma} \boxtimes \overline{\sigma} \in I_{r+}^{\boxplus}$

σ \ $\overline{\sigma}$	$I_{r+}^{\boxplus}$	$(I_{r+}^{\boxplus})^c$
$I_{r+}^{\boxplus}$	$I_{r+}^{\boxplus}$	**EXIST**
$(I_{r+}^{\boxplus})^c$	**NOT EXIST**	**EXIST**

Back to classical case

$$\begin{cases} Z \sim N(0,1), E \sim \mathrm{Exp}(1/2), \\ A \sim \text{symm. arcsine distri. on } (-1,1) \\ Z, E, A, V\text{: indep.}, V > 0 \end{cases}$$

$\mathcal{L}(Z) = \mathcal{L}(E^{1/2}A)$
$\Rightarrow \mathcal{L}(VZ) = \mathcal{L}(E^{1/2}VA)$
$\Rightarrow \mathcal{L}((VZ)^2) = \mathcal{L}(EV^2A^2)$
Mixture of Exponential distri. is classical ID.
$\Rightarrow \mathcal{L}((VZ)^2)$ is ID.

Ex. (Type W)

- standard S-C Distri. : $\mathbf{w} = \mathbf{w} \boxtimes \delta_1$
- Symm. free stable distri. (including Cauchy distri.).
- Symm. beta Distri. with parameter $(1/2, 3/2)$

Symm. beta Distri. with parameter
$(1/2, 3/2) \notin \Lambda(\{\textbf{Type } G\})$

$$\mathbf{b_1} = \mathbf{w} \boxtimes (\mathbf{a}^{+} \boxtimes \overline{\mathbf{m_2}}) \in \{\textbf{Type W}\}$$

$\mathbf{a}^{+}$**: arcsine distri on** $(0, 1)$.
$\mathbf{m_2} = \overline{\mathbf{m_2}} \boxtimes \overline{\mathbf{m_2}}$
$\mathbf{b_1}(dx) = \frac{1}{2\pi}|x|^{-1/2}(2 - |x|)^{1/2}\mathbb{1}_{(-2,2)}dx$

§4 On free regular infinitely divisible distributions

N. Sakuma.
On free regular infinitely divisible distributions.
Bessatsu, B27, 115–122, (2011).
O. Arizmendi, T. Hasebe, and N. Sakuma.
Free regular infinite divisibility and squares of random variables with $\boxplus$-infinitely divisible distributions.
ALEA, 5, 271–291, (2013).

Closure properties

Thm. (Arizmendi, Hasebe and S)

$\mu, \nu \in I_{r+}^{\boxplus}$, $\sigma \in I^{\boxplus} \cap \mathcal{P}_s$.

(1) $\mu \boxtimes \nu \in I_{r+}^{\boxplus}$.

(2) $\mu^{\boxtimes t} \in I_{r+}^{\boxplus}$ $(t \geq 1)$.

(3) $\mu \boxtimes \sigma \in I^{\boxplus}$.

Ex.

The condition $\mu, \nu \in I_{r+}^{\boxplus}$ is essential.

w_+: S-C distri. with mean $\mathbf{2}$ and variance $\mathbf{1}$.

$\mathbf{w}_+ \boxtimes \mathbf{w}_+ \notin I^{\boxplus}$

$\boxplus$-ID for distributions

Thm. (Arizmendi, Hasebe and S)

$\mathcal{L}(X) \in I^{\boxplus} \cap \mathcal{P}_s \Rightarrow \mathcal{L}(X^2) \in I_{r+}^{\boxplus}$.

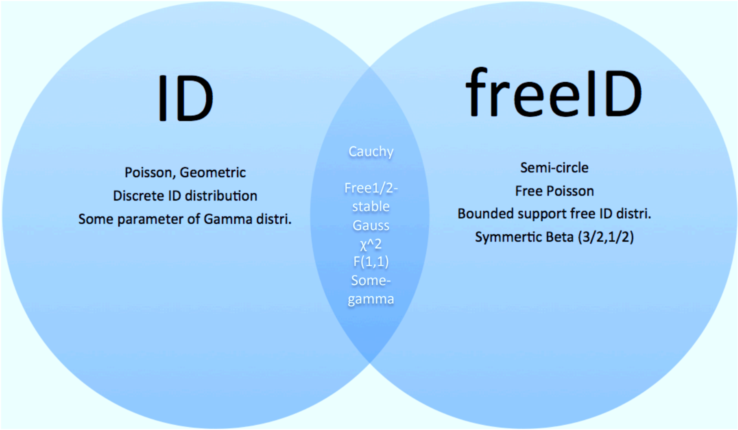
In particular, $\mathcal{L}(X^2)$ is compound free Poisson.

Thm. (Belinschi, Bożejko, Lehner and Speicher)

Gauss distri. is $\boxplus$-ID.

Thm. (Arizmendi, Hasebe and S)

- χ^2 distri. is $\boxplus$-ID.
- $F(1,1)$ distri. is $\boxplus$-ID.


ID
Poisson, Geometric
Discrete ID distribution
Some parameter of Gamma distri.
Cauchy
Free1/2-
stable
Gauss
χ^2
F(1,1)
Some-
gamma
freeID
Semi-circle
Free Poisson
Bounded support free ID distri.
Symmertic Beta (3/2,1/2)