On parameter estimation for critical affine processes

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Affine semigroup

Definition (Duffie et al., 2003): A transition semigroup $(P_t)_{t\in\mathbb{R}_+}$ with state space $\mathbb{R}_+\times\mathbb{R}^d$ is called an affine semigroup if its characteristic function has the representation

$$\int_{\mathbb{R}_+\times\mathbb{R}^d} \mathrm{e}^{\langle u,\xi\rangle} P_t(x,\mathrm{d}\xi) = \mathrm{e}^{\langle x,\psi(t,u)\rangle+\phi(t,u)}$$

for $x\in\mathbb{R}_+\times\mathbb{R}^d$, $u\in U$ and $t\in\mathbb{R}_+$, where

- $U:=\{z_1+iz_2 : z_1\in\mathbb{R}_-, z_2\in\mathbb{R}\}\times(i\mathbb{R}^d)$,
- $\psi(t,\cdot)=(\psi_1(t,\cdot),\psi_2(t,\cdot))$ is a continuous $\mathbb{C}^{1+d}$-valued function on U,
- $\phi(t,\cdot)$ is a continuous $\mathbb{C}$-valued function on U with $\phi(t,0)=0$.

(Here $\mathbb{R}_+:=[0,\infty)$ and $\mathbb{R}_-:=(-\infty,0]$.)

Admissible parameters

A parameter set (a,α,b,β,m,μ) is called admissible if

- $a,\alpha\in\mathbb{R}^{(1+d)\times(1+d)}$ are sym. positive semidef. matrices with $a_{1,1}=0$, and $b\in\mathbb{R}_+\times\mathbb{R}^d$,
- $\beta\in\mathbb{R}^{(1+d)\times(1+d)}$ with $\beta_{1,j}=0$ for all $j\in\{2,\dots,1+d\}$,
- $m(\mathrm{d}\xi)=m(\mathrm{d}\xi_1,\mathrm{d}\xi_2)$ is a σ-finite measure on $\mathbb{R}_+\times\mathbb{R}^d$ supported by $(\mathbb{R}_+\times\mathbb{R}^d)\setminus\{(0,0)\}$ such that

$$\int_{\mathbb{R}_+\times\mathbb{R}^d}\left[\xi_1+(\|\xi_2\|\wedge\|\xi_2\|^2)\right]m(\mathrm{d}\xi)<\infty,$$

- $\mu(\mathrm{d}\xi)=\mu(\mathrm{d}\xi_1,\mathrm{d}\xi_2)$ is a σ-finite measure on $\mathbb{R}_+\times\mathbb{R}^d$ supported by $(\mathbb{R}_+\times\mathbb{R}^d)\setminus\{(0,0)\}$ such that

$$\int_{\mathbb{R}_+\times\mathbb{R}^d}\|\xi\|\wedge\|\xi\|^2\mu(\mathrm{d}\xi)<\infty.$$

A scaling theorem for affine processes

Theorem: For all $\vartheta>0$, let $(Y^{(\vartheta)}(t),X^{(\vartheta)}(t))_{t\in\mathbb{R}_+}$ be an affine process with state space $\mathbb{R}_+\times\mathbb{R}^d$ and with admissible parameters $(a^{(\vartheta)},\alpha^{(\vartheta)},b^{(\vartheta)},\beta^{(\vartheta)},m,\mu)$ such that additionally

$$\int_{\mathbb{R}_+\times\mathbb{R}^d}\|\xi\|\,m(\mathrm{d}\xi)<\infty \quad\text{and}\quad \int_{\mathbb{R}_+\times\mathbb{R}^d}\|\xi\|^2\mu(\mathrm{d}\xi)<\infty.$$

Let $a,\alpha,\beta\in\mathbb{R}^{(1+d)\times(1+d)}$, $b\in\mathbb{R}_+\times\mathbb{R}^d$, and let $(Y(t),X(t))_{t\in\mathbb{R}_+}$ be an affine process with state space $\mathbb{R}_+\times\mathbb{R}^d$ and with admissible parameters $(a,\widetilde{\alpha},\widetilde{b},\beta,0,0)$, where

$$\widetilde{\alpha}:=\alpha+\frac{1}{2}\int_{\mathbb{R}_+\times\mathbb{R}^d}\xi\xi^\top\mu(\mathrm{d}\xi),$$

and $\widetilde{b}=(\widetilde{b}_i)_{i=1}^{1+d}$ with $\widetilde{b}_i:=b_i$ for $i\in\{2,\dots,1+d\}$ and

$$\widetilde{b}_1:=b_1+\int_{\mathbb{R}_+\times\mathbb{R}^d}\xi_1\,m(\mathrm{d}\xi).$$

If

$$\vartheta^{-1}a^{(\vartheta)}\to a,\quad \alpha^{(\vartheta)}\to\alpha,\quad b^{(\vartheta)}\to b,\quad \vartheta\beta^{(\vartheta)}\to\beta,$$

$$\vartheta^{-1}(Y^{(\vartheta)}(0),X^{(\vartheta)}(0))\xrightarrow{\mathcal{L}}(Y(0),X(0))$$

as $\vartheta\to\infty$, then

$$\left(\vartheta^{-1}Y^{(\vartheta)}(\vartheta t),\vartheta^{-1}X^{(\vartheta)}(\vartheta t)\right)_{t\in\mathbb{R}_+}\xrightarrow{\mathcal{L}}(Y(t),X(t))_{t\in\mathbb{R}_+}$$

in $\mathrm{D}(\mathbb{R}_+,\mathbb{R}_+\times\mathbb{R}^d)$ as $\vartheta\to\infty$.

The limit process $(Y(t),X(t))_{t\in\mathbb{R}_+}$ has continuous sample paths a.s.

An affine two-factor model

For $a>0$, $b,\vartheta\in\mathbb{R}_+$ and $m\in\mathbb{R}$, let us consider the SDE

$$\begin{cases}\mathrm{d}Y_t=(a-bY_t)\,\mathrm{d}t+\sqrt{Y_t}\,\mathrm{d}W_t,\\ \mathrm{d}X_t=(m-\vartheta X_t)\,\mathrm{d}t+\sqrt{Y_t}\,\mathrm{d}B_t,\end{cases}\qquad t\in\mathbb{R}_+,$$

where $(W_t)_{t\in\mathbb{R}_+}$ and $(B_t)_{t\in\mathbb{R}_+}$ are independent standard Wiener processes.

Note that Y is a Cox-Ingersol-Ross (CIR) process.

Classification

If $\mathbb{E}(Y_0)<\infty$ and $\mathbb{E}(|X_0|)<\infty$, then for any $t\in\mathbb{R}_+$,

$$\begin{bmatrix}\mathbb{E}(Y_t)\\ \mathbb{E}(X_t)\end{bmatrix}=\begin{bmatrix}\mathrm{e}^{-bt} & 0\\ 0 & \mathrm{e}^{-\vartheta t}\end{bmatrix}\begin{bmatrix}\mathbb{E}(Y_0)\\ \mathbb{E}(X_0)\end{bmatrix}+\begin{bmatrix}\int_0^t\mathrm{e}^{-bs}\,\mathrm{d}s & 0\\ 0 & \int_0^t\mathrm{e}^{-\vartheta s}\,\mathrm{d}s\end{bmatrix}\begin{bmatrix}a\\ m\end{bmatrix}.$$

We call $(Y_t,X_t)_{t\in\mathbb{R}_+}$ subcritical, critical or supercritical if the spectral radius of the matrix $\mathrm{diag}(\mathrm{e}^{-bt},\mathrm{e}^{-\vartheta t})$ is less than 1, equal to 1 or greater than 1, resp. Equivalently,

subcritical if $b>0$ and $\vartheta>0$,
critical if $b=0$, $\vartheta\geq 0$ or $b\geq 0$, $\vartheta=0$,
supercritical if $b<0$ or $\vartheta<0$.

LSE of (ϑ,m)

The LSE of (ϑ,m) based on observations $X_0,X_1,\dots,X_n$ can be obtained by solving the extremum problem

$$(\widehat{\vartheta}_n,\widehat{m}_n):=\arg\min_{(\vartheta,m)\in\mathbb{R}^2}\sum_{i=1}^n(X_i-X_{i-1}-(m-\vartheta X_{i-1}))^2.$$

One can check that

$$\widehat{\vartheta}_n=-\frac{n\sum_{i=1}^n(X_i-X_{i-1})X_{i-1}-\sum_{i=1}^nX_{i-1}\sum_{i=1}^n(X_i-X_{i-1})}{n\sum_{i=1}^nX_{i-1}^2-\left(\sum_{i=1}^nX_{i-1}\right)^2},$$

and

$$\widehat{m}_n=\frac{\sum_{i=1}^nX_{i-1}^2\sum_{i=1}^n(X_i-X_{i-1})-\sum_{i=1}^nX_{i-1}\sum_{i=1}^n(X_i-X_{i-1})X_{i-1}}{n\sum_{i=1}^nX_{i-1}^2-\left(\sum_{i=1}^nX_{i-1}\right)^2}$$

provided that $n\sum_{i=1}^nX_{i-1}^2-\left(\sum_{i=1}^nX_{i-1}\right)^2>0$.

Limit behaviour of $(\widehat{\vartheta}_n,\widehat{m}_n)$ in a critical case

If $(b,\vartheta)=(0,0)$, $\mathbb{P}(Y_0\geq 0)=1$, $\mathbb{E}(Y_0)<\infty$, and $\mathbb{E}(X_0^2)<\infty$, then $\mathbb{P}\left(n\sum_{i=1}^nX_{i-1}^2-\left(\sum_{i=1}^nX_{i-1}\right)^2>0\right)=1$, $n\geq 2$, and there exists a unique LSE $(\widehat{\vartheta}_n,\widehat{m}_n)$ having the forms given above. Further,

$$n\widehat{\vartheta}_n\xrightarrow{\mathcal{L}}-\frac{\int_0^1\mathcal{X}_t\,\mathrm{d}\mathcal{X}_t-\mathcal{X}_1\int_0^1\mathcal{X}_t\,\mathrm{d}t}{\int_0^1\mathcal{X}_t^2\,\mathrm{d}t-\left(\int_0^1\mathcal{X}_t\,\mathrm{d}t\right)^2}\quad\text{as } n\to\infty,$$

and

$$\widehat{m}_n\xrightarrow{\mathcal{L}}\frac{\mathcal{X}_1\int_0^1\mathcal{X}_t^2\,\mathrm{d}t-\int_0^1\mathcal{X}_t\,\mathrm{d}t\int_0^1\mathcal{X}_t\,\mathrm{d}\mathcal{X}_t}{\int_0^1\mathcal{X}_t^2\,\mathrm{d}t-\left(\int_0^1\mathcal{X}_t\,\mathrm{d}t\right)^2}\quad\text{as } n\to\infty,$$

where $(\mathcal{X}_t)_{t\in\mathbb{R}_+}$ is the second coordinate of the two-dimensional affine process $(\mathcal{Y}_t,\mathcal{X}_t)_{t\in\mathbb{R}_+}$ given by the unique strong solution of the SDE

$$\begin{cases}\mathrm{d}\mathcal{Y}_t=a\,\mathrm{d}t+\sqrt{\mathcal{Y}_t}\,\mathrm{d}\mathcal{W}_t,\\ \mathrm{d}\mathcal{X}_t=m\,\mathrm{d}t+\sqrt{\mathcal{Y}_t}\,\mathrm{d}\mathcal{B}_t,\end{cases}\qquad t\in\mathbb{R}_+,$$

with initial value $(\mathcal{Y}_0,\mathcal{X}_0)=(0,0)$, where $(\mathcal{W}_t)_{t\in\mathbb{R}_+}$ and $(\mathcal{B}_t)_{t\in\mathbb{R}_+}$ are independent standard Wiener processes.

References

[1] M. Barczy, L. Döring, Z. Li and G. Pap: On parameter estimation for critical affine processes. *Electronic Journal of Statistics* 7 (2013) 647–696.