Structural properties and exponential ergodicity for equations with Lévy noise

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Wroclaw - 7th Conference on Lévy Processes - July 2013

A motivating example

Consider the following semilinear parabolic SPDE in a bounded domain $D \subset \mathbb{R}^d$ with regular boundary ∂D:

$$\begin{cases} dX(t,\xi) = [\Delta X(t,\xi) + F(X(t,\xi))]dt + dZ_t(\xi), \\ X(0,\xi) = x(\xi),\ \ \xi \in D, \\ X(t,\xi) = 0,\ \ \xi \in \partial D, \end{cases} \tag{1}$$

The Laplace operator $-\Delta$ with the Dirichlet boundary condition has a discrete spectrum $\{\gamma_k\}$ in $H = L^2(D)$. We denote by $\{e_k\}$ the basis consisting of its normalised eigenfunctions.

If for simplicity take $D = [0,\pi]^d$ then

$$e_k(\xi) = \left(\frac{2}{\pi}\right)^{\frac{d}{2}} \sin(k_1\xi_1)\cdots\sin(k_d\xi_d), \quad k \in \mathbb{N}^d,\ \xi \in D.$$

It is easy to see that $\Delta e_k = -|k|^2 e_k$, i.e.

$$\gamma_k = |k|^2 = k_1^2 + \ldots + k_d^2.$$

We consider $Z = (Z_t) = (Z(t, \cdot))$ as a cylindrical α-stable noise of the form

$$Z_t(\xi) = \sum_{k=1}^{\infty} \beta_k \, z_k(t) \, e_k(\xi), \quad t \geqslant 0, \quad \xi \in D,$$

where $\{z_k(t)\}_k$ are i.i.d. real symmetric α-stable processes with $\alpha \in (0, 2)$, i.e.,

$$E[e^{i\lambda z_k(t)}] = e^{-t|\lambda|^{\alpha}},$$

$t \geqslant 0, \lambda \in \mathbb{R}, k \geqslant 1$.

Moreover β_k are positive numbers (for instance, $\beta_k = 1, k \geqslant 1$).

We want to study the dynamics defined by (1) in the Hilbert space $H = L^2(D)$. Assuming that

$$F : H \to H \quad \text{is Lipschitz continuous and bounded}$$

we will give conditions on (β_n) and (γ_n) such that there exists a unique invariant measure μ and we have **exponential convergence to μ in the total variation norm.**

Precisely if $\alpha \in (1, 2)$ we prove that there exists $\mu \in \mathcal{P}(H)$ so that for any $p \in (0, \alpha)$, we have (where $X_t^x = X^x(t, \cdot)$ starts from $x \in H$)

$$\|\mathcal{L}aw\,(X_t^x) - \mu\|_{\mathrm{TV}} \leqslant Ce^{-ct}\,(1 + |x|^p)\,, \quad t \geqslant 0, \tag{2}$$

where C and c depends on $(p, \alpha, \|F\|_0, \beta = (\beta_k)$ and $\gamma = (\gamma_k)$; we have set $|x| = |x|_H$.

Recall the *total variation distance:* if $\mu_1, \mu_2 \in \mathcal{P}(H)$ then

$$\|\mu_1 - \mu_2\|_{\mathrm{TV}} = \frac{1}{2} \sup_{\substack{f \in B_b(H) \\ \|f\|_0 = 1}} |\mu_1(f) - \mu_2(f)| = \sup_{\Gamma \in \mathcal{B}(H)} |\mu_1(\Gamma) - \mu_2(\Gamma)|.$$

where $\mu(f) = \int_H f\, d\mu, f \in B_b(H)$ (i.e., $f : H \to \mathbb{R}$ is Borel and bounded); $\|\cdot\|_0$ is the supremum norm.

We write our SDE in the abstract form

$$dX_t = [AX_t + F(X_t)]dt + dZ_t, \quad X_0 = x \in H, \tag{3}$$

where H is a given real separable Hilbert space with basis (e_k).

Hypothesis

(A1) A is a dissipative (self-adjoint) operator defined by

$$A = \sum_{k \geqslant 1} (-\gamma_k) e_k \otimes e_k$$

with $0 < \gamma_1 \leqslant \gamma_2 \leqslant \ldots \leqslant \gamma_k \leqslant \ldots$ and $\gamma_k \to \infty$ as $k \to \infty$.

(A2) Z_t is a cylindrical α-stable process with $Z_t = \sum_{k \geqslant 1} \beta_k z_k(t) e_k$ where $\{z_k(t)\}_{k \geqslant 1}$ are i.i.d. symmetric α-stable processes with $0 < \alpha < 2$ and β_k are positive constants such that, for some $\epsilon \in (0, 1)$,

$$\sum_{k \geqslant 1} \frac{\beta_k^\alpha}{\gamma_k^{1-\alpha\epsilon}} < \infty$$

(A3) $F : H \to H$ is Lipschitz and bounded.

(A4) There exist some $\theta \in (0, 1)$ and $C > 0$ so that $\beta_k \geqslant C \gamma_k^{-\theta + 1/\alpha}$ (if $\beta_k \equiv 1$ then $\theta = \frac{1}{\alpha}$ and $\alpha > 1$).

Basic properties

$$D(A) = \{x = (x_n) \in H \,:\, \sum_{n\geqslant 1} x_n^2\gamma_n^2 < +\infty\}.$$

In addition A generates a compact C_0-semigroup (e^{tA}) on H such that

$$e^{tA}e_k = e^{-\gamma_k t}e_k, \quad k \in \mathbb{N}, \quad t \geqslant 0.$$

If $F = 0$ our equation is an infinite sequence of independent one dimensional stochastic equations, i.e.,

$$dX_t^n = -\gamma_n X_t^n dt + \beta_n dZ_t^n, \quad X_0^n = x_n, \;\; n \in \mathbb{N},$$

with $x = (x_n) \in l^2 \sim H$, i.e.

$$X_t^n = e^{-\gamma_n t}x_n + \int_0^t e^{-\gamma_n(t-s)}\beta_n dZ_s^n, \quad n \in \mathbb{N}, \;\; t \geqslant 0$$

A predictable H-valued stochastic process $X = (X_t^x)$, depending on $x \in H$, is a *mild solution* if, for any $t \geqslant 0$, $x \in H$, it holds ($P - a.s.$):

$$X_t^x = e^{tA}x + \int_0^t e^{(t-s)A} F(X_s^x) ds + Z_A(t), \quad \text{with}$$

$$Z_A(t) = \int_0^t e^{(t-s)A} dZ_s = \sum_{n \geqslant 1} \Big(\int_0^t e^{-\gamma_n (t-s)} \beta_n dZ_s^n \Big) e_n.$$

Theorem (P. Zabczyk PTRF 2011)

Assume $\alpha \in (0,2)$ and (A1), (A2) and (A3) (even with $\epsilon = 0$) and $x \in H$.

There exists a unique mild solution $X^x = (X_t^x)$. Moreover,

(i) X^x is stochastically continuous;

(ii) for any $x \in H$, $T > 0$, X^x has trajectories in $L^p(0,T;H)$, for any $0 < p < \alpha$, P-a.s.;

(iii) $X^x = (X_t^x)$ is a Markov-Feller process.

Let us denote by P_t the transition semigroup associated to X^x, i.e. $P_t f(x) = E[f(X_t^x)]$.

Moreover for $\mu \in \mathcal{P}(H)$ we set $(P_t^* \mu)(f) = \int_H (P_t f)(x) \mu(dx), f \in B_b(H)$.

Some references

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$$\partial_t u = u_{xx} - uu_x + \xi_t$$

The main convergence result

Theorem (P.-Xu-Shirikyan-Zabczyk 2012)

Let $\alpha \in (1,2)$. There exists $\mu \in \mathcal{P}(H)$ so that for any $p \in (0,\alpha)$ and any measure $\nu \in \mathcal{P}(H)$ with finite p^{th} moment, we have

$$\|P_t^* \nu - \mu\|_{\mathrm{TV}} \leqslant C e^{-ct} \left(1 + \int_H |x|^p \nu(dx)\right), \quad t \geqslant 0, \tag{4}$$

where $C = C(p, \alpha, \theta, \beta, \gamma, \epsilon, \|F\|_0)$ and $c = c(p, \alpha, \theta, \beta, \gamma, \epsilon, \|F\|_0)$.

Remark

Let us come back to the stochastic semilinear equation on $D = [0,\pi]^d$ with $d \geqslant 1$

$$\begin{cases} dX(t,\xi) = [\Delta X(t,\xi) + F(X(t,\xi))]dt + dZ_t(\xi), \\ X(0,\xi) = x(\xi), \\ X(t,\xi) = 0, \quad \xi \in \partial D, \end{cases}$$

Δ with Dirichlet B.C.

Consider the dynamics in $H = L^2(D)$ with basis of eigenfunctions $\{e_k\}_{k\in\mathbb{N}^d}$ and

$$Z_t = \sum_{k\in\mathbb{N}^d} |k|^\beta z_k(t) e_k$$

where $\{z_k(t)\}_k$ are i.i.d. symmetric α-stable processes with $\alpha \in (0,2)$ and $\beta \in \mathbb{R}$. Note that

$$\sum_{k\in\mathbb{N}^d} \frac{|k|^{\beta\alpha}}{|k|^2} < \infty$$

if and only if $2 > d + \alpha\beta$.

From the main result we get

If F is a bounded Lipschitz function and

$$2 > d + \alpha\beta, \ \ \frac{1}{\alpha} - \frac{\beta}{2} < 1,$$

or equivalently, $\frac{d}{\alpha} < \frac{2}{\alpha} - \beta < 2$, then the system is exponentially ergodic.

Structural properties of the SPDEs

To prove the convergence result we also need

Theorem (P. - Zabczyk 2011)

Let us consider the unique mild solution (X_t^x) *(this form a Markov process with the transition semigroup* P_t*).*

The process is irreducible.

Moreover, if $\alpha \in (1,2)$*, there exists* $C > 0$ *such that*

$$|P_t f(x) - P_t f(y)| \leqslant \frac{C\|f\|_0}{t^{1/\theta} \wedge 1} |x-y|, \quad x, y \in H, \ t > 0 \tag{5}$$

where θ *is given in (A4).*

Estimate (5) implies the *uniform strong Feller property.*

Irreducibility means that for any $t_0 > 0$ we have

$$P_{t_0}(x, A) > 0$$

for any $x \in H$ and A non-empty open set ($P_{t_0}(x, \cdot)$ is the transition probability of X^x at $t = t_0$).

The starting point of the proof of the convergence result

- The existence of an invariant measure can be established by a compactness argument (see [P.-Xu-Zabczyk2011]). Also uniqueness follows from irreducibility and strong Feller property.

- One can prove that any invariant measure μ has finite p^{th} moment ($p < \alpha$):

$$m_p(\mu) := \int_H |x|^p \mu(dx) < \infty \quad \text{for any } p \in (0, \alpha).$$

To this purpose one can use that for all $t > 0$ and $n \in \mathbb{N}$, we have

$$E\big(|X_t^x|^p \wedge n\big) \leqslant \big(C_p e^{-p\gamma_1 t}|x|^p\big) \wedge n + C, \tag{6}$$

where $C = C(\alpha, \beta, \gamma, p, \|F\|_0)$. Integrating this inequality against $\mu(dx)$, we get easily the assertion by passing to the limit as $n \to \infty$.

A useful inequality

To prove the exponential convergence it suffices to show that for some $T > 0$

$$\|P_{kT}(x_1,\cdot) - P_{kT}(x_2,\cdot)\|_{\mathrm{TV}} \leqslant C\,(1 + |x_1|^p + |x_2|^p)e^{-ckT}, \quad x_1, x_2 \in H, \;\; k \geqslant 1, \tag{7}$$

where

$$P_{kT}(x_1) = P^*_{kT}(\delta_{x_1}) \;\; \text{is the law of} \;\; X^{x_1}_{KT}$$

C and c are positive constants not depending on x_1, x_2, and k.

Indeed, if (7) is established, then for any measures $\nu_1, \nu_2 \in \mathcal{P}(H)$ with finite p^{th} moment we derive

$$\|P^*_{kT}\nu_1 - P^*_{kT}\nu_2\|_{\mathrm{TV}} \leqslant C\,\big(1 + m_p(\nu_1) + m_p(\nu_2)\big)e^{-ckT}, \quad k \in \mathbb{N}. \tag{8}$$

(this implies, in particular, that an invariant measure is unique).

Writing any $t \geqslant 0$ in the form $t = kT + s$ with $0 \leqslant s < T$ one obtains (see also (6))

$$\begin{aligned}
\|P^*_t\nu_1 - P^*_t\nu_2\|_{\mathrm{TV}} &= \|P^*_{kT}(P^*_s\nu_1) - P^*_{kT}(P^*_s\nu_2)\|_{\mathrm{TV}} \\
&\leqslant C\,\big(1 + m_p(P^*_s\nu_1) + m_p(P^*_s\nu_2)\big)e^{-ckT} \\
&\leqslant C_1\big(1 + m_p(\nu_1) + m_p(\nu_2)\big)e^{-ct}.
\end{aligned}$$

How to get (7)?

To prove (7) and so the exponential ergodicity there are two approaches.

The first one is by adapting a classical Harris'theorem (1956).

The other is by a coupling argument which goes back to Doeblin (1940).

An idea on the Doeblin method (more intuition but more technically involved)

Let us fix a large $T > 0$ and consider the restriction of the Markov process (X_t^x), $x \in H$, to the times kT proportional to T, $k \geqslant 1$.

We denote by $(Y_k) =$ the resulting discrete-time Markov process, by $\mathbb{P}_x$ the corresponding family of probability measures, and by $P_k(x, \Gamma)$ the transition function.

By the structural properties of the SPDEs if initial points $x_1, x_2 \in H$ are such that $|x_1 - x_2| \leqslant r$, with a sufficiently small $r > 0$, then

$$\|P_1(x_1, \cdot) - P_1(x_2, \cdot)\|_{\mathrm{TV}} \leqslant \frac{1}{2}. \tag{9}$$

An idea on the coupling method by Doeblin

Now let (Y_k^1, Y_k^2) be a homogeneous discrete-time Markov process in the extended phase space $H \times H$ such that the following properties hold for the pair (Y_1^1, Y_1^2) under the law $\mathbb{P}_{(x_1,x_2)}$ corresponding to the initial point (x_1, x_2):

(a) The laws of Y_1^1 and Y_1^2 coincide with $P_1(x_1, \cdot)$ and $P_1(x_2, \cdot)$, respectively.

(b) If $\max(|x_1|, |x_2|) > r$ and $x_1 \neq x_2$, then the random variables Y_1^1 and Y_1^2 are independent.

(c) If $\max(|x_1|, |x_2|) \leqslant r$ and $x_1 \neq x_2$, then

$$P_{(x_1,x_2)}\{Y_1^1 \neq Y_1^2\} = \|P_1(x_1, \cdot) - P_1(x_2, \cdot)\|_{\mathrm{TV}}.$$

(d) If $x_1 = x_2$, then $Y_1^1 = Y_1^2$ with probability 1.

Such chain can be constructed with the help of maximal coupling of measures.

A pair of random variables (ξ_1, ξ_2) defined on the same probability space is called a *coupling* for μ_1 and $\mu_2 \in P(H)$ if

$$D(\xi_i) = \mu_i$$

for $i = 1, 2$, where $D(\cdot)$ denotes the distribution. A coupling (ξ_1, ξ_2) is *maximal* if

$$P\{\xi_1 \neq \xi_2\} = \|\mu_1 - \mu_2\|_{\mathrm{TV}},$$

and the random variable ξ_1 and ξ_2 conditioned on the event $N := \{\xi_1 \neq \xi_2\}$ are independent (see Lindvall, Lectures on the Coupling method, 2002).

Now let us construct an auxiliary Markov chain in $H \times H$.

Let $T > 0$ be some fixed real number to be chosen later. For any $x := (x_1, x_2) \in H \times H$,

$$M(x) = (M_1(x), M_2(x))$$

is the maximal coupling of $(P_T)^* \delta_{x_1}$ and $(P_T)^* \delta_{x_2}$. Let us define a transition function $\tilde{P}_T(x, \cdot)$ on $H \times H$ such that

$$\tilde{P}_T(x; A_1 \times A_2) = \begin{cases} P_T(x_1, A_1 \cap A_2) \ \text{ if } \ x_1 = x_2, \\ D(M_1(x), M_2(x))(A_1 \times A_2) \ \text{ if } x_1, x_2 \in B(r) \text{ with } x_1 \neq x_2, \\ P_T(x_1, A_1) P_T(x_2, A_2) \quad \text{otherwise,} \qquad A_1, A_2 \in \mathcal{B}(H), \end{cases}$$

$P_T(x_i, \cdot)$ is the transition probability of $X_T^{x_i}$ for $i = 1, 2$.

Combining properties (a)–(d) with irreducibility of (Y_k) and inequality (9), it is possible to prove that the coupling stopping time

$$\rho = \min\{k \geqslant 0 : Y_k^1 = Y_k^2\}$$

is $\mathbb{P}_{(x_1,x_2)}$ -almost surely finite and has a finite exponential moment, i.e.,

$$\mathbb{P}_{(x_1,x_2)}\{\rho > k\} \leqslant Ce^{-\eta kT}(1 + |x_1|^p + |x_2|^p), \tag{10}$$

where $x = (x_1, x_2) \in H \times H$ is arbitrary, and the positive constants η and C do not depend on x.

Now it follows from (d) that $Y_k^1 = Y_k^2$ for $k \geqslant \rho$. We can thus write

$$|P_k(x_1, \Gamma) - P_k(x_2, \Gamma)| = |\mathbb{E}_{(x_1,x_2)}(I_\Gamma(Y_k^1) - I_\Gamma(Y_k^2))| \tag{11}$$

$$\leqslant \mathbb{E}_{(x_1,x_2)}\left(1_{\{\rho>k\}}\left|1_\Gamma\left(Y_k^1\right) - 1_\Gamma\left(Y_k^2\right)\right|\right) \leqslant \mathbb{P}_{(x_1,x_2)}\{\rho > k\},$$

where $\Gamma \subset H$ is an arbitrary Borel subset and I_Γ stands for its indicator function.

Finally, by taking the supremum over Γ, with constants independent of x_1, x_2, and k,

$$\|P_{kT}(x_1, \cdot) - P_{kT}(x_2, \cdot)\|_{\mathrm{TV}} \leqslant C\,(1 + |x_1|^p + |x_2|^p)e^{-\eta kT}, \quad x_1, x_2 \in H. \tag{12}$$

The Harris approach

We concentrate on the Harris approach. For a surprisingly short and nice proof of the next resul we refer to Hairer's lecture notes

Theorem

Let P_t be a Markov semigroup in the Polish space X such that there exists $T_0 > 0$ and measurable $V : X \to \mathbb{R}_+$ which satisfies:

(i) there exists $\gamma < 1$ and $K > 0$ such that $P_{T_0}V(x) \leqslant \gamma V(x) + K$, $x \in X$.

(ii) for every $R > 0$ there exists $\delta > 0$ such that

$$\|P^*_{T_0}\delta_x - P^*_{T_0}\delta_y\|_{TV} \leqslant 2 - \delta,$$

for all $x, y \in X$ such that $V(x) + V(y) \leqslant R$.
Then there exist some $T > 0$ and $\beta < 1$ such that

$$\int_X (1 + V(x))|P^*_T\mu - P^*_T\nu|(dx) \leqslant \beta \int_X (1 + V(x))|\mu - \nu|(dx).$$

The key point for Harris' theorem approach is to guess the Liapunov function V and to check conditions (i) and (ii).

Note that the desired estimate (7) holds if we are able to apply the Harris theorem to equation (3) with

$$V(x) = |x|^p$$

and $p \in (0, \alpha)$. Indeed, once this is done, we obtain that there exists $T > 0$ such that

$$\begin{aligned}
\|P_{kT}(x_1, \cdot) - P_{kT}(x_2, \cdot)\|_{TV} &\leqslant \int_H (1 + V(x))|P^*_{kT}\delta_{x_1} - P^*_{kT}\delta_{x_2}|(dx) \\
&\leqslant \beta^k \int_H (1 + V(x))|\delta_{x_1} - \delta_{x_2}|(dx) \\
&\leqslant 2\beta^k \big(1 + |x_1|^p + |x_2|^p\big), \quad k \geqslant 1.
\end{aligned}$$

This immediately implies (7).

Checking conditions (i) and (ii) of the Harris theorem

Define, for any $\epsilon \geqslant 0$,

$$H^{\epsilon} = \left\{ x = \sum_{k \geqslant 1} x_k e_k \in H : \sum_{k \geqslant 1} \gamma_k^{2\epsilon} |x_k|^2 < \infty \right\}.$$

Note that H^{ϵ} coincides with the domain of $(-A)^{\epsilon}$ and that $H^0 = H$. Set

$$| \cdot |_{\epsilon} = | \cdot |_{H_{\epsilon}}.$$

For $x \in H^{\epsilon}$ and $R > 0$, $B_{\epsilon}(x, R)$ is the closed ball in H^{ϵ} of radius R centered at x. We also write $B_{\epsilon}(R) := B_{\epsilon}(0, R)$ and $B(x, R) := B_0(x, R)$.

Choosing $V(x) = |x|^p$ with $p \in (0, \alpha)$ and applying the following lemma one gets (i).

Lemma

Let (X_t^x) be the solution to Eq. (3) with $x \in H^{\epsilon}$. For any $p \in (0, \alpha)$, there exist some constants $C_1 = C_1(p) > 0$ and $C_2 = C_2(p, \epsilon, \gamma, \beta, \|F\|_0) > 1$ such that

$$E|X_t^x|_{\epsilon}^p \leqslant C_1 e^{-p\gamma_1 t} |x|_{\epsilon}^p + C_2, \quad \forall\, t > 0,$$

where $C_1(p) \leqslant 1$ for $p \in (0, 1]$ and $C_1(p) = 3^{p-1}$ otherwise.

To prove condition (ii), we shall use the *irreducibility and uniform strong Feller property*. Recall irreducibility:

for any $t > 0$ and $B(y, r)$ with arbitrary $y \in H$ and $r > 0$, we have

$$P\left(X_t^x \in B(y, r)\right) > 0. \tag{13}$$

Let us prove (ii).

Let x and y satisfy $|x|^p + |y|^p \leqslant R$. One can prove that, for any fixed $T_0 > 0$,

$$E[|X_{T_0}^x|_\epsilon^p] + E[|X_{T_0}^y|_\epsilon^p] \leqslant C(|x|^p + |y|^p + 1) \leqslant C_1.$$

It follows that there exists some $R_1 > 0$ such that

$$P\left(|X_{T_0}^x|_\epsilon \leqslant R_1\right) > 1/2, \;\; P\left(|X_{T_0}^y|_\epsilon \leqslant R_1\right) > 1/2.$$

Since $\gamma_k \to \infty$, $B_\epsilon(R_1)$ is compact in H. By (13), for any $r > 0$ we have some $\delta(r) > 0$ such that

$$\inf_{x \in B_\epsilon(R_1)} P\left(X_{T_0}^x \in B(r)\right) \geqslant 2\delta.$$

By Markov property and the above three inequalities, we have

$$P\left(X^x_{2T_0} \in B(r)\right) > \delta, \;\; P\left(X^y_{2T_0} \in B(r)\right) > \delta.$$

Without loss of generality, in the next computations we assume that X^x_t and X^y_t are independent

By Markov property and the uniform strong Feller property

$$\begin{aligned}
&\|P^*_{3T_0}\delta_x - P^*_{3T_0}\delta_y\|_{TV} \sup_{\|\phi\|_0 \leqslant 1} |E[P_{T_0}\phi(X^x_{2T_0}) - P_{T_0}\phi(X^y_{2T_0})]| \\
&\leqslant P\{X^x_{2T_0} \notin B(r)\} + P\{X^y_{2T_0} \notin B(r)\} \\
&\;+ E\left\{\sup_{\|\phi\|_0 \leqslant 1} |P_{T_0}\phi(X^x_{2T_0}) - P_{T_0}\phi(X^y_{2T_0})|, \; X^x_{2T_0} \in B(r), X^y_{2T_0} \in B(r)\right\} \\
&\leqslant 2 - P\{X^x_{2T_0} \in B(r)\} - P\{X^y_{2T_0} \in B(r)\} + 2CrP\{X^x_{2T_0} \in B(r)\} P\{X^y_{2T_0} \in B(r)\} \\
&\leqslant 2 - \delta
\end{aligned} \tag{14}$$

as $r > 0$ sufficiently small. This completes the proof.

In [P. Xu Shirikyan Zabczyk 12] we also consider the SDE

$$X_t = x + \int_0^t AX_s \mathrm{d}s + \int_0^t b(X_s)\,\mathrm{d}s + L_t, \quad x \in \mathbb{R}^d, \;\; t \geqslant 0. \tag{15}$$

When the $d \times d$-matrix $A = 0$, **pathwise uniqueness** for (15) has been proved in [P. Osaka J. Math, 2012] assuming

(H1) $b : \mathbb{R}^d \to \mathbb{R}^d$ is bounded and β-Hölder continuous, (i.e. $b \in C_b^\beta(\mathbb{R}^d, \mathbb{R}^d)$; $\|b\|_\beta = \|b\|_0 + [b]_\beta < +\infty$),

$$[b]_\beta = \sup_{x \neq y,\, x,y \in \mathbb{R}^d} \frac{|b(x) - b(y)|}{|x-y|^\beta}, \quad \beta \in (0,1).$$

(H2) $L = (L_t)$ is a non-degenerate d-dimensional symmetric α-stable Lévy process, $d \geqslant 1$.

(H3) $\alpha \in [1,2)$ and $\beta + \dfrac{\alpha}{2} > 1$.

In [P. Xu Shirikyan Zabczyk 12] we prove **exponential ergodicity** assuming that all eigenvalues of A have negative real parts and also $\alpha > 1$.

A (positive) Borel measure γ on $\mathbb{R}^d$ is symmetric if $\gamma(D) = \gamma(-D)$, $D \in \mathcal{B}(\mathbb{R}^d)$.

Let $\alpha \in (0,2)$. In the SDE we consider a d-dimensional Lévy process $L = (L_t)$ which is also *symmetric α-stable.* This means that

$$E[e^{i\langle L_t,h\rangle}] = e^{-t\psi(h)}, \quad \psi(h) = -\int_{\mathbb{R}^d} \Big(e^{i\langle h,y\rangle} - 1 - i\langle h,y\rangle\, 1_{\{|h|\leqslant 1\}}(y)\Big)\nu(dy), \qquad (16)$$

$h \in \mathbb{R}^d$, $t \geqslant 0$, where for $D \subset \mathbb{R}^d$, 1_D is the indicator function of D ($1_D(x) = 1 \iff x \in D$) and ν is a Borel measure (the Lévy measure of L) such that

$$\nu(D) = \int_S \mu(d\xi) \int_0^\infty 1_D(r\xi)\frac{dr}{r^{1+\alpha}}, \quad D \in \mathcal{B}(\mathbb{R}^d), \qquad (17)$$

for some symmetric, non-zero finite Borel measure μ concentrated on $S = \{y \in \mathbb{R}^d \,:\, |y| = 1\}$.

Formula (16) is the Lévy-Khintchine formula for L. By (17), we rewrite (16) as

$$\psi(h) = -\int_{\mathbb{R}^d} \big(\cos(\langle h,y\rangle) - 1\big)\nu(dy) = c_\alpha \int_S |\langle h,\xi\rangle|^\alpha \mu(d\xi), \;\; h \in \mathbb{R}^d.$$

The measure μ is called the spectral measure of L.

We assume that *the stable process L is not degenerate*; this means that (see [Sztonik 2010])

The support of the spectral measure μ is not contained in a proper linear subspace of $\mathbb{R}^d$.

It is not difficult to show that this is equivalent to

$$\psi(u) \geqslant C_\alpha |u|^\alpha, \quad u \in \mathbb{R}^d,$$

for some positive constant C_α.