One dimensional completely asymmetric Markov processes

Pierre Patie

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Setting

Let $X = (X_t)_{t\geq 0}$ be a $E \subseteq \mathbb{R}$-valued standard process, defined on the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$, i.e.

- $t \mapsto X_t$ is càdlàg $\mathbb{P}$-a.s.
- X has the strong Markov property
- X is quasi-left continuous on $[0, \zeta)$

where $\zeta = \inf\{t \geq 0;\ X_t = \Delta\}$ is the lifetime of X and Δ is the cemetery point.

We assume that $\mathfrak{h} \notin E$ and is non-entrance whith $\mathfrak{h}$ the right endpoint of E.

We write $\forall x \in E, \mathbb{P}_x(X_0 = x) = 1$. We denote by $(P_t)_{t\geq 0}$ its semigroup, i.e. for positive borelian function f with $f(\Delta) = 0$,

$$P_t f(x) = \mathbb{E}_x\left[f(X_t)\mathbb{I}_{\{t<\zeta\}}\right] = \mathbb{E}_x\left[f(X_t)\right].$$

For a set $A \subseteq \overline{E}$, we write

$$T_A = \inf\{t > 0;\ X_t \in A\},$$

and simply $T_y = T_{\{y\}}$.

Moreover we assume :

A) $\mathbb{P}(X_{t-} \geq X_t, \forall 0 \leq t < \zeta) = 1$ (no positive jumps).

B) $\forall x, y \in E$, $\mathbb{P}_x(T_y < \zeta) > 0$ (visit points).

X is called a completely asymmetric Markov process (for short **CAMP**).

Spectrally negative Lévy processes, continuous state branching processes with immigration and several generalizations of the classical Ornstein-Uhlenbeck process are CAMP.

Remark : Our results extend to skip-free Markov chains.

We want to study the following questions :

1. Can we characterize in terms of fundamental functions the law of the first exit time of the interval (a, b), $a, b \in E$, i.e $T_{(a,b)^c}$, for the CAMP ?

2. Do CAMP admit a resolvent density ? Can we describe it in terms of fundamental functions ?

If X is a diffusion on E, then Feller (52,54) showed that, with $q > 0$,

$$\mathbb{E}_x\left[e^{-qT_y}\right] = \begin{cases} \frac{H_q^{\uparrow}(x)}{H_q^{\uparrow}(y)}, & x \leq y, \\ \frac{H_q^{\downarrow}(x)}{H_q^{\downarrow}(y)}, & x \geq y, \end{cases}$$

where $H_q^{\uparrow}$ (resp. $H_q^{\downarrow}$) is the fundamental increasing (resp. decreasing) solution to the second order differential equation subject to appropriate boundary conditions, associated to the infinitesimal generator $\mathbf{L}$ of X,

$$\mathbf{L}f_q(x) := \sigma^2(x)f_q''(x) + \mu(x)f_q'(x) = qf_q(x) \tag{1}$$

where σ, μ are smooth functions.

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He also showed that

$$u_q(x, y) = w_q^{-1} H_q^{\uparrow}(x \wedge y) H_q^{\downarrow}(x \vee y)$$

where $U_q f(x) = \int_E u_q(x, y)f(y)m(dy)$ for some positive measure m and w_q is the Wronskian.

1. Around spectrally negative Lévy processes : Spitzer (57), Takàcs (66), Emery (73), Suprun (76), Bertoin (97), Kyprianou and Palmowski (05), Doney (08) ...
2. Generalized spectrally negative Ornstein-Uhlenbeck processes : Hadjiev (83), Novikov (04, 08), Jacobsen and Jensen (07).
3. Spectrally negative positive self-similar Markov and related processes : P. (08).

Futher references on Markov processes and potential theory :

- Blumenthal and Getoor (68), Dellacherie and Meyer (83), Sharpe (88), Chung and Walsh (10) ...
- Doob (57), Kunita and Watanabe (65), Smythe and Walsh (73), Bally and Stoica (92), Fitzsimmons and Getoor (06,09).

Classification of points

Let $x \in E$, we say that :

- x is oscillating if $\mathbb{P}_x$-a.s. $T_{(x,\infty)} = 0$ and $T_{(-\infty,x)} = 0$.
- x is climbing if $\mathbb{P}_x$-a.s. $T_{(x,\infty)} = 0$ and $T_{(-\infty,x)} > 0$.

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Proposition

All points in $int(E)$ are either oscillating or climbing.

Let $\mathcal{S}_q, q > 0$, be the set of q-excessive functions, i.e. $f \geq 0$ Borelian s.t.

$$e^{-qt} P_t f(x) \leq f(x), \ \forall x \in E$$

with $\lim_{t \downarrow 0} e^{-qt} P_t f(x) = f(x)$.

Corollary

If $f \in \mathcal{S}_q$ then f is continuous (resp. right-continuous) at oscillating (resp. climbing) points.

The q-fundamental excessive function H_q

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We observe from the strong Markov property and the absence of positive jumps that, for any $x \vee o < a < y$, the mapping

$$y \mapsto \frac{\mathbb{E}_x\left[e^{-qT_y}\right]}{\mathbb{E}_o\left[e^{-qT_y}\right]} = \frac{\mathbb{E}_x\left[e^{-qT_a}\right]\mathbb{E}_a\left[e^{-qT_y}\right]}{\mathbb{E}_o\left[e^{-qT_a}\right]\mathbb{E}_a\left[e^{-qT_y}\right]} = \frac{\mathbb{E}_x\left[e^{-qT_a}\right]}{\mathbb{E}_o\left[e^{-qT_a}\right]}$$

is constant on $(x \vee o, \mathfrak{h})$.

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$$H_q(x) = \lim_{y \to \mathfrak{h}} \frac{\mathbb{E}_x\left[e^{-qT_y}\right]}{\mathbb{E}_o\left[e^{-qT_y}\right]}.$$

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Moreover, let $x < y$ and choose $y \vee o < a$, then as above

$$H_q(x) = \frac{\mathbb{E}_x\left[e^{-qT_a}\right]}{\mathbb{E}_o\left[e^{-qT_a}\right]} = \frac{\mathbb{E}_x\left[e^{-qT_y}\right]\mathbb{E}_y\left[e^{-qT_a}\right]}{\mathbb{E}_o\left[e^{-qT_a}\right]} = \mathbb{E}_x\left[e^{-qT_y}\right] H_q(y).$$

Let $o \in E$ be a reference point and $q > 0$.

Proposition

There exists a q-excessive function H_q which is positive, continuous and increasing on E with $H_q(o) = 1$.

Moreover, we have the simple hitting time formula :

$$\mathbb{E}_x\left[e^{-qT_y}\right] = \frac{H_q(x)}{H_q(y)}, \quad x < y.$$

H_q is called the q-fundamental excessive function of X or P.

Let P^{H_q} be the semigroup defined, for all $t \geq 0$, by

$$P_t^{H_q} f(x) = \frac{e^{-qt}}{H_q(x)} P_t H_q f(x), \quad x \in E,$$

that is the excessive Doob-H_q transform of the semigroup $(e^{-qt}P_t)_{t\geq 0}$. We write $\mathbb{P}^{H_q}$ for the law of its standard realization.

Proposition

1. H_q is the unique minimal excessive function such that $\mathbb{P}^{H_q}\left(X_{\zeta^-} = \mathfrak{h}\right) = 1$ and $H_q(o) = 1$.
2. H_q is either q-invariant, i.e. $\forall t \geq 0, \forall x \in E$,

$$e^{-qt} P_t H_q(x) = H_q(x)$$

or q-purely excessive, i.e. $\forall x \in E$,

$$\lim_{t\to\infty} e^{-qt} P_t H_q(x) = 0.$$

The process killed at time $T_{(-\infty,b)}$

Let $b \in E$ and $P^{b)}$ be the subordinate semigroup defined, for all $t \geq 0$, by

$$P_t^{b)} f(x) = \mathbb{E}_x \left[f(X_t) \mathbb{I}_{\{t < T_{(-\infty,b)}\}} \right].$$

Its realization is a CAMP.

Corollary

There exists a q-fundamental excessive function for $P^{b)}$, denoted by $H_q^{b)}$, which is positive, continuous and increasing on (b, ∞) with $H_q^{b)}(b) = 0$ (resp. > 0) if b is oscillating (resp. climbing) and $H_q^{b)} = \mathbf{0}_{(-\infty,b)}$. Moreover, we have

$$\mathbb{E}_x \left[e^{-qT_y} \mathbb{I}_{\{T_y < T_{(-\infty,b)}\}} \right] = \frac{H_q^{b)}(x)}{H_q^{b)}(y)}, \quad x < y.$$

The general hitting time formula

Proposition

We have

$$K_q^{b)}(\mathfrak{h}) = \lim_{x \to \mathfrak{h}} \frac{H_q(x)}{H_q^{b)}(x)} \in (0, \infty).$$

Moreover, for any $x \geq b$,

$$\mathbb{E}_x\left[e^{-qT_b}\right] = \frac{1}{H_q(b)}\left(H_q(x) - K_q^{b)}(\mathfrak{h}) H_q^{b)}(x)\right).$$

Since $\mathbb{P}^{H_q}\left(X_{\zeta-}=\mathfrak{h}\right)=1$, we have, for any $b<x<a\in E$,

$$\mathbb{P}_x^{H_q}\left(T_{(-\infty,b)}<T_a\right)+\mathbb{P}_x^{H_q}\left(T_a<T_{(-\infty,b)}\right)=1 \tag{2}$$

and

$$\mathbb{P}_x^{H_q}\left(T_{(-\infty,b)}<T_a\right)=\mathbb{P}_x^{H_q}\left(T_b<T_a\right).$$

On the other hand, we have

$$\begin{aligned}\mathbb{P}_x^{H_q}\left(T_a<T_{(-\infty,b)}\right) &= \frac{H_q(a)}{H_q(x)}\mathbb{E}_x\left[e^{-qT_a}\mathbb{I}_{\{T_a<T_{(-\infty,b)}\}}\right]=\frac{H_q(a)}{H_q(x)}\frac{H_q^{b)}(x)}{H_q^{b)}(a)},\\ \mathbb{P}_x^{H_q}\left(T_b<T_a\right) &= \frac{H_q(b)}{H_q(x)}\mathbb{E}_x\left[e^{-qT_b}\mathbb{I}_{\{T_b<T_a\}}\right].\end{aligned}$$

Thus rearranging the terms in (2), we obtain

$$\mathbb{E}_x\left[e^{-qT_b}\mathbb{I}_{\{T_b<T_a\}}\right]=\frac{1}{H_q(b)}\left(H_q(x)-K_q^{b)}(a)H_q^{b)}(x)\right).$$

To conclude we let $a\uparrow\mathfrak{h}$.

Proposition

1. There exists an excessive measure ξ, i.e. $\xi P_t f \leq \xi f$, such that, for all $q > 0$, there exits a positive, locally bounded and jointly measurable function u_q such that

$$\forall x \in E, \quad U_q(x, \mathrm{d}y) = u_q(x, y)\xi(\mathrm{d}y) \tag{3}$$

where U_q is the kernel of the q-resolvent of P.

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2. Moreover, we have the hitting-resolvent identity : for any $x, y \in E$,

$$\mathbb{E}_x\left[e^{-qT_y}\right] = \begin{cases} \frac{u_q(x,y)}{u_q(y,y)} & \text{if } y \text{ is oscillating,} \\ \frac{u_q(x,y)}{u_q(y_-,y)} & \text{otherwise,} \end{cases} \tag{4}$$

where $u_q(y_-, y) = \lim_{x \uparrow y} u_q(x, y)$.

We aim to make use of the identity (valid when $\mathbb{P}(T_{(-\infty,b)} < \zeta) = 1$)

$$\mathbb{E}_x\left[e^{-qT_{(-\infty,b)}}\right] = 1 - q\int_b^{\mathfrak{h}} u_q^{b)}(x,y)\xi(dy), \quad x \geq b, \tag{5}$$

where $u_q^{b)}$ is the q-resolvent density associated to $P_t^{b)}$.

Since ξ is excessive, there exists a dual process $\widehat{X}$, which is a left-continuous moderate Markov process and its resolvent $\widehat{U}_q$ satisfies

$$\langle U_q f, g\rangle_\xi = \langle f, \hat{U}_q g\rangle_\xi, \quad q > 0.$$

Proposition

The process $\hat{X}$ has no negative jumps and visits points below. Then, there exists a co-excessive function $\widehat{H}_q$ which is right-continuous, decreasing such that $\widehat{H}_q(o) = 1$ and

$$\widehat{\mathbb{E}}_y\left[e^{-qT_x}\right] = \frac{\widehat{H}_q(y)}{\widehat{H}_q(x)} = \frac{u_q(x,y)}{u_q(x,x_+)}, \quad y > x.$$

Using the previous Proposition together with the identities (4), we can describe the q-resolvent u_q. Using a moderate version of the Hunt's switching identity we get the representation of $u_q^{b)}$ as follows.

Proposition

Let $q > 0$ and write $C_q(o) = u_q(o, o_+)$. Then, for any $x, y \in E$,

$$u_q(x,y) \quad = \quad C_q(o)\widehat{H}_q(y)\left(H_q(x) - K_q^{y)}(\mathfrak{h})\, H_q^{y)}(x)\right). \tag{6}$$

For any $x, y > b$,

$$u_q^{b)}(x,y) \quad = \quad C_q(o)\widehat{H}_q(y)\left(K_q^{b)}(\mathfrak{h})H_q^{b)}(x) - K_q^{y)}(\mathfrak{h})\, H_q^{y)}(x)\right).$$

We can now use the last expression with the identity (5) to get a representation of $\mathbb{E}_x\left[e^{-qT_{(-\infty,b)}}\right]$.

Spectrally negative Lévy processes

Let X be a spectrally negative Lévy process. For any $x \in \mathbb{R}$,

$$P_t e_\lambda(x) = e^{\psi(\lambda)t+\lambda x}, \quad \lambda \geq 0,$$

where $e_\lambda(x) = e^{\lambda x}$ and ψ is the Lévy-Khintchine exponent of X.

1. Let $\phi : [0,\infty) \to [\phi(0),\infty)$ such that $\psi(\phi(q)) = q$. Then

   $$e^{-qt} P_t e_{\phi(q)}(x) = e_{\phi(q)}(x), \quad q, t \geq 0.$$

   Hence, with $o = 0$, we have $H_q(x) = e^{\phi(q)x}$.
2. $\xi(dy) = dy$, $\hat{X} \stackrel{d}{=} -X$ and $\hat{H}_q(y) = e^{-\phi(q)y}$.
3. Next, observe, for any $0 < \lambda < \phi(q)$, that

$$\int_{\mathbb{R}} e^{-\lambda x} u_q(x,0)dx \quad = \quad \int_0^\infty e^{-qt} P_t e_\lambda(0)dt = \frac{-1}{\psi(\lambda) - q}.$$

4. Thus, from (6), we deduce that

$$\int_0^\infty e^{-\lambda x} C_q(0)\left(H_q(x) - K_q^{0)}(\mathfrak{h}) H_q^{0)}(x)\right) dx \quad = \quad \frac{-1}{\psi(\lambda)-q} - \frac{C_q(0)}{\phi(q)-\lambda}.$$

On the one hand, by a principle of analytical continuation, we obtain

$$C_q(0) = \lim_{\lambda\to\phi(q)} \frac{\lambda-\phi(q)}{\psi(\lambda)-q} = \frac{1}{\psi'(\phi(q))} = \phi'(q).$$

On the other hand, one may set $K_q^{0)}(\mathfrak{h}) = \frac{1}{C_q(0)}$ to get, for any $\lambda > 0$,

$$\int_0^\infty e^{-\lambda x} H_q^{0)}(x) dx \quad = \quad \frac{1}{\psi(\lambda)-q}.$$

4. After some easy computations, using the identity (5), we deduce that for any $x \geq 0$,

$$\mathbb{E}_x[e^{-qT_{(-\infty,0)}}] = 1 + q\int_0^x H_q^{0)}(y) dy - \frac{q}{\phi(q)} H_q^{0)}(x).$$