

One dimensional completely asymmetric Markov processes

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Setting

Let $X = (X_t)_{t \geq 0}$ be a $E \subseteq \mathbb{R}$ -valued standard process, defined on the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$, i.e.

- $t \mapsto X_t$ is càdlàg \mathbb{P} -a.s.
- X has the strong Markov property
- X is quasi-left continuous on $[0, \zeta)$

where $\zeta = \inf\{t \geq 0; X_t = \Delta\}$ is the lifetime of X and Δ is the cemetery point.

We assume that $\mathfrak{h} \notin E$ and is non-entrance with \mathfrak{h} the right endpoint of E .

We write $\forall x \in E, \mathbb{P}_x(X_0 = x) = 1$. We denote by $(P_t)_{t \geq 0}$ its semigroup, i.e. for positive borelian function f with $f(\Delta) = 0$,

$$P_t f(x) = \mathbb{E}_x [f(X_t) \mathbb{I}_{\{t < \zeta\}}] = \mathbb{E}_x [f(X_t)].$$

For a set $A \subseteq \bar{E}$, we write

$$T_A = \inf\{t > 0; X_t \in A\},$$

and simply $T_y = T_{\{y\}}$.

Moreover we assume :

A) $\mathbb{P}(X_{t-} \geq X_t, \forall 0 \leq t < \zeta) = 1$ (no positive jumps).

B) $\forall x, y \in E, \mathbb{P}_x(T_y < \zeta) > 0$ (visit points).

X is called a completely asymmetric Markov process (for short **CAMP**). Spectrally negative Lévy processes, continuous state branching processes with immigration and several generalizations of the classical Ornstein-Uhlenbeck process are CAMP.

Remark : Our results extend to skip-free Markov chains.

We want to study the following questions :

- 1 Can we characterize in terms of fundamental functions the law of the first exit time of the interval (a, b) , $a, b \in E$, i.e $T_{(a,b)^c}$, for the CAMP ?
- 2 Do CAMP admit a resolvent density ? Can we describe it in terms of fundamental functions ?

If X is a **diffusion** on E , then Feller (52,54) showed that, with $q > 0$,

$$\mathbb{E}_x \left[e^{-qT_y} \right] = \begin{cases} \frac{H_q^\uparrow(x)}{H_q^\uparrow(y)}, & x \leq y, \\ \frac{H_q^\downarrow(x)}{H_q^\downarrow(y)}, & x \geq y, \end{cases}$$

where H_q^\uparrow (resp. H_q^\downarrow) is the fundamental increasing (resp. decreasing) solution to **the second order differential equation** subject to appropriate boundary conditions, associated to the infinitesimal generator \mathbf{L} of X ,

$$\mathbf{L}f_q(x) := \sigma^2(x)f_q''(x) + \mu(x)f_q'(x) = qf_q(x) \quad (1)$$

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He also showed that

$$u_q(x, y) = w_q^{-1} H_q^\uparrow(x \wedge y) H_q^\downarrow(x \vee y)$$

where $U_q f(x) = \int_E u_q(x, y) f(y) m(dy)$ for some positive measure m and w_q is the Wronskian.

- 1 Around spectrally negative Lévy processes : Spitzer (57), Takács (66), Emery (73), Suprun (76), Bertoin (97), Kyprianou and Palmowski (05), Doney (08) ...
- 2 Generalized spectrally negative Ornstein-Uhlenbeck processes : Hadjiev (83), Novikov (04, 08), Jacobsen and Jensen (07).
- 3 Spectrally negative positive self-similar Markov and related processes : P. (08).

Further references on Markov processes and potential theory :

- Blumenthal and Gettoor (68), Dellacherie and Meyer (83), Sharpe (88), Chung and Walsh (10) ...
- Doob (57), Kunita and Watanabe (65), Smythe and Walsh (73), Bally and Stoica (92), Fitzsimmons and Gettoor (06,09).

Classification of points

Let $x \in E$, we say that :

- x is **oscillating** if \mathbb{P}_x -a.s. $T_{(x,\infty)} = 0$ and $T_{(-\infty,x)} = 0$.
- x is **climbing** if \mathbb{P}_x -a.s. $T_{(x,\infty)} = 0$ and $T_{(-\infty,x)} > 0$.

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Proposition

All points in $\text{int}(E)$ are either oscillating or climbing.

Let $\mathcal{S}_q, q > 0$, be the set of q -excessive functions, i.e. $f \geq 0$ Borelian s.t.

$$e^{-qt} P_t f(x) \leq f(x), \forall x \in E$$

with $\lim_{t \downarrow 0} e^{-qt} P_t f(x) = f(x)$.

Corollary

If $f \in \mathcal{S}_q$ then f is continuous (resp. right-continuous) at oscillating (resp. climbing) points.

The q -fundamental excessive function H_q

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We observe from the strong Markov property and the absence of positive jumps that, for any $x \vee o < a < y$, the mapping

$$y \mapsto \frac{\mathbb{E}_x [e^{-qT_y}]}{\mathbb{E}_o [e^{-qT_y}]} = \frac{\mathbb{E}_x [e^{-qT_a}] \mathbb{E}_a [e^{-qT_y}]}{\mathbb{E}_o [e^{-qT_a}] \mathbb{E}_a [e^{-qT_y}]} = \frac{\mathbb{E}_x [e^{-qT_a}]}{\mathbb{E}_o [e^{-qT_a}]}$$

is constant on $(x \vee o, \mathfrak{h})$.

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$$H_q(x) = \lim_{y \rightarrow \mathfrak{h}} \frac{\mathbb{E}_x [e^{-qT_y}]}{\mathbb{E}_o [e^{-qT_y}]}.$$

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Moreover, let $x < y$ and choose $y \vee o < a$, then as above

$$H_q(x) = \frac{\mathbb{E}_x [e^{-qT_a}]}{\mathbb{E}_o [e^{-qT_a}]} = \frac{\mathbb{E}_x [e^{-qT_y}] \mathbb{E}_y [e^{-qT_a}]}{\mathbb{E}_o [e^{-qT_a}]} = \mathbb{E}_x [e^{-qT_y}] H_q(y).$$

Let $o \in E$ be a reference point and $q > 0$.

Proposition

There exists a q -excessive function H_q which is positive, continuous and increasing on E with $H_q(o) = 1$.

Moreover, we have the **simple hitting time formula** :

$$\mathbb{E}_x \left[e^{-qT_y} \right] = \frac{H_q(x)}{H_q(y)}, \quad x < y.$$

H_q is called the **q -fundamental excessive function** of X or P .

Let P^{H_q} be the semigroup defined, for all $t \geq 0$, by

$$P_t^{H_q} f(x) = \frac{e^{-qt}}{H_q(x)} P_t H_q f(x), \quad x \in E,$$

that is the excessive Doob- H_q transform of the semigroup $(e^{-qt} P_t)_{t \geq 0}$. We write \mathbb{P}^{H_q} for the law of its standard realization.

Proposition

- 1 H_q is the **unique minimal excessive function** such that $\mathbb{P}^{H_q}(X_{\zeta^-} = \mathfrak{h}) = 1$ and $H_q(o) = 1$.
- 2 H_q is either **q -invariant**, i.e. $\forall t \geq 0, \forall x \in E$,

$$e^{-qt} P_t H_q(x) = H_q(x)$$

or **q -purely excessive**, i.e. $\forall x \in E$,

$$\lim_{t \rightarrow \infty} e^{-qt} P_t H_q(x) = 0.$$

The process killed at time $T_{(-\infty, b)}$

Let $b \in E$ and P^b be the subordinate semigroup defined, for all $t \geq 0$, by

$$P_t^b f(x) = \mathbb{E}_x \left[f(X_t) \mathbb{I}_{\{t < T_{(-\infty, b)}\}} \right].$$

Its realization is a CAMP.

Corollary

There exists a q -fundamental excessive function for P^b , denoted by H_q^b , which is positive, continuous and increasing on (b, ∞) with $H_q^b(b) = 0$ (resp. > 0) if b is oscillating (resp. climbing) and $H_q^b = \mathbf{0}_{(-\infty, b)}$.

Moreover, we have

$$\mathbb{E}_x \left[e^{-qT_y} \mathbb{I}_{\{T_y < T_{(-\infty, b)}\}} \right] = \frac{H_q^b(x)}{H_q^b(y)}, \quad x < y.$$

The general hitting time formula

Proposition

We have

$$K_q^{(b)}(h) = \lim_{x \rightarrow h} \frac{H_q(x)}{H_q^{(b)}(x)} \in (0, \infty).$$

Moreover, for any $x \geq b$,

$$\mathbb{E}_x \left[e^{-qT_b} \right] = \frac{1}{H_q(b)} \left(H_q(x) - K_q^{(b)}(h) H_q^{(b)}(x) \right).$$

Since $\mathbb{P}^{H_q}(X_{\zeta^-} = \mathfrak{h}) = 1$, we have, for any $b < x < a \in E$,

$$\mathbb{P}_x^{H_q}(T_{(-\infty, b)} < T_a) + \mathbb{P}_x^{H_q}(T_a < T_{(-\infty, b)}) = 1 \quad (2)$$

and

$$\mathbb{P}_x^{H_q}(T_{(-\infty, b)} < T_a) = \mathbb{P}_x^{H_q}(T_b < T_a).$$

On the other hand, we have

$$\mathbb{P}_x^{H_q}(T_a < T_{(-\infty, b)}) = \frac{H_q(a)}{H_q(x)} \mathbb{E}_x \left[e^{-qT_a} \mathbb{I}_{\{T_a < T_{(-\infty, b)}\}} \right] = \frac{H_q(a)}{H_q(x)} \frac{H_q^b(x)}{H_q^b(a)},$$

$$\mathbb{P}_x^{H_q}(T_b < T_a) = \frac{H_q(b)}{H_q(x)} \mathbb{E}_x \left[e^{-qT_b} \mathbb{I}_{\{T_b < T_a\}} \right].$$

Thus rearranging the terms in (2), we obtain

$$\mathbb{E}_x \left[e^{-qT_b} \mathbb{I}_{\{T_b < T_a\}} \right] = \frac{1}{H_q(b)} \left(H_q(x) - K_q^b(a) H_q^b(x) \right).$$

To conclude we let $a \uparrow \mathfrak{h}$.

Proposition

- 1 There exists an excessive measure ξ , i.e. $\xi P_t f \leq \xi f$, such that, for all $q > 0$, there exists a positive, locally bounded and jointly measurable function u_q such that

$$\forall x \in E, \quad U_q(x, dy) = u_q(x, y)\xi(dy) \quad (3)$$

where U_q is the kernel of the q -resolvent of P .

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- ② Moreover, we have the hitting-resolvent identity : for any $x, y \in E$,

$$\mathbb{E}_x \left[e^{-qT_y} \right] = \begin{cases} \frac{u_q(x, y)}{u_q(y, y)} & \text{if } y \text{ is oscillating,} \\ \frac{u_q(x, y)}{u_q(y-, y)} & \text{otherwise,} \end{cases} \quad (4)$$

where $u_q(y-, y) = \lim_{x \uparrow y} u_q(x, y)$.

We aim to make use of the identity (valid when $\mathbb{P}(T_{(-\infty, b)} < \zeta) = 1$)

$$\mathbb{E}_x \left[e^{-qT_{(-\infty, b)}} \right] = 1 - q \int_b^{\cdot} u_q^{(b)}(x, y) \xi(dy), \quad x \geq b, \quad (5)$$

where $u_q^{(b)}$ is the q -resolvent density associated to $P_t^{(b)}$.

Since ξ is excessive, there exists a **dual process** \hat{X} , which is a **left-continuous moderate Markov process** and its resolvent \hat{U}_q satisfies

$$\langle U_q f, g \rangle_\xi = \langle f, \hat{U}_q g \rangle_\xi, \quad q > 0.$$

Proposition

The process \hat{X} has no negative jumps and visits points below. Then, there exists a co-excessive function \hat{H}_q which is right-continuous, decreasing such that $\hat{H}_q(o) = 1$ and

$$\hat{\mathbb{E}}_y \left[e^{-qT_x} \right] = \frac{\hat{H}_q(y)}{\hat{H}_q(x)} = \frac{u_q(x, y)}{u_q(x, x_+)}, \quad y > x.$$

Using the previous Proposition together with the identities (4), we can describe the q -resolvent u_q . Using a moderate version of the Hunt's switching identity we get the representation of u_q^b as follows.

Proposition

Let $q > 0$ and write $C_q(o) = u_q(o, o_+)$. Then, for any $x, y \in E$,

$$u_q(x, y) = C_q(o) \widehat{H}_q(y) \left(H_q(x) - K_q^y(\mathfrak{h}) H_q^y(x) \right). \quad (6)$$

For any $x, y > b$,

$$u_q^b(x, y) = C_q(o) \widehat{H}_q(y) \left(K_q^b(\mathfrak{h}) H_q^b(x) - K_q^y(\mathfrak{h}) H_q^y(x) \right).$$

We can now use the last expression with the identity (5) to get a representation of $\mathbb{E}_x \left[e^{-qT_{(-\infty, b)}} \right]$.

Spectrally negative Lévy processes

Let X be a spectrally negative Lévy process. For any $x \in \mathbb{R}$,

$$P_t e_\lambda(x) = e^{\psi(\lambda)t + \lambda x}, \quad \lambda \geq 0,$$

where $e_\lambda(x) = e^{\lambda x}$ and ψ is the Lévy-Khintchine exponent of X .

1. Let $\phi : [0, \infty) \rightarrow [\phi(0), \infty)$ such that $\psi(\phi(q)) = q$. Then

$$e^{-qt} P_t e_{\phi(q)}(x) = e_{\phi(q)}(x), \quad q, t \geq 0.$$

Hence, with $o = 0$, we have $H_q(x) = e^{\phi(q)x}$.

2. $\xi(dy) = dy$, $\hat{X} \stackrel{d}{=} -X$ and $\hat{H}_q(y) = e^{-\phi(q)y}$.

3. Next, observe, for any $0 < \lambda < \phi(q)$, that

$$\int_{\mathbb{R}} e^{-\lambda x} u_q(x, 0) dx = \int_0^\infty e^{-qt} P_t e_\lambda(0) dt = \frac{-1}{\psi(\lambda) - q}.$$

4. Thus, from (6), we deduce that

$$\int_0^\infty e^{-\lambda x} C_q(0) \left(H_q(x) - K_q^{(0)}(h) H_q^{(0)}(x) \right) dx = \frac{-1}{\psi(\lambda) - q} - \frac{C_q(0)}{\phi(q) - \lambda}.$$

On the one hand, by a principle of analytical continuation, we obtain

$$C_q(0) = \lim_{\lambda \rightarrow \phi(q)} \frac{\lambda - \phi(q)}{\psi(\lambda) - q} = \frac{1}{\psi'(\phi(q))} = \phi'(q).$$

On the other hand, one may set $K_q^{(0)}(h) = \frac{1}{C_q(0)}$ to get, for any $\lambda > 0$,

$$\int_0^\infty e^{-\lambda x} H_q^{(0)}(x) dx = \frac{1}{\psi(\lambda) - q}.$$

4. After some easy computations, using the identity (5), we deduce that for any $x \geq 0$,

$$\mathbb{E}_x[e^{-qT_{(-\infty, 0]}}] = 1 + q \int_0^x H_q^{(0)}(y) dy - \frac{q}{\phi(q)} H_q^{(0)}(x).$$