APPROXIMATION OF MARKOV SEMIGROUPS VIA FEYNMAN FORMULAE

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Concepts

Let $(X, \|\cdot\|_X)$ be a Banach space, $\mathcal{L}(X)$ be the space of all continuous linear operators on X equipped with the strong operator topology, $\|\cdot\|$ be the operator norm on $\mathcal{L}(X)$ and Id be the identity operator in X. If $\mathrm{Dom}(L) \subset X$ is a linear subspace and $L : \mathrm{Dom}(L) \to X$ is a linear operator, then $\mathrm{Dom}(L)$ denotes the domain of L. A one-parameter family $(T_t)_{t\geq 0}$ of bounded linear operators $T_t : X \to X$ is called a C_0-semigroup, if $T_0 = \mathrm{Id}$, $T_{s+t} = T_s \circ T_t$ for all $s, t \geq 0$ and $\lim_{t\to 0} \|T_t\varphi - \varphi\|_X = 0$ for all $\varphi \in X$. If $(T_t)_{t\geq 0}$ is a C_0-semigroup on a Banach space $(X, \|\cdot\|_X)$, then the generator L of $(T_t)_{t\geq 0}$ is defined by

$$L\varphi := \lim_{t\to 0} \frac{T_t\varphi - \varphi}{t}, \quad \mathrm{Dom}(L) := \left\{ \varphi \in X \,\middle|\, \lim_{t\to 0} \frac{T_t\varphi-\varphi}{t} \text{ exists as a strong limit} \right\}.$$

Consider an evolution equation $\frac{\partial f}{\partial t} = Lf$. If L is the generator of a C_0-semigroup $(T_t)_{t\geq 0}$ (denoted also as e^{tL}) on $(X, \|\cdot\|_X)$, then the (mild) solution of the Cauchy problem for this equation with the initial value $f(0) = f_0 \in X$ is given by $f(t) = T_t f_0$ for all $f_0 \in X$. In probabilistic setting: L is the generator of a Markov process $(\xi_t)_{t\geq 0}$ with transitional probability $P_t(q, dy)$, i.e. $T_t f_0(q) = \int f_0(y) P_t(q, dy) = \mathbb{E}^q[f_0(\xi_t)]$. Therefore, to solve the evolution equation $\frac{\partial f}{\partial t} = Lf$ means to construct a semigroup $(T_t)_{t\geq 0}$ with the given generator L and, hence, to approximate the transitional probability of the underlying process. If the desired semigroup is not known explicitly it can be approximated. One of the tools to approximate semigroups is based on the Chernoff theorem.

Theorem 1 (Chernoff). *Let X be a Banach space, $F : [0, \infty) \to \mathcal{L}(X)$ be a (strongly) continuous mapping such that $F(0) = \mathrm{Id}$ and $\|F(t)\| \leq e^{at}$ for some $a \in [0, \infty)$ and all $t \geq 0$. Let D be a linear subspace of $\mathrm{Dom}(F'(0))$ such that the restriction of the operator $F'(0)$ to this subspace is closable. Let $(L, \mathrm{Dom}(L))$ be this closure. If $(L, \mathrm{Dom}(L))$ is the generator of a strongly continuous semigroup $(T_t)_{t\geq 0}$, then for any $t_0 > 0$ the sequence $(F(t/n))^n)_{n\in\mathbb{N}}$ converges to $(T_t)_{t\geq 0}$ as $n \to \infty$ in the strong operator topology, uniformly with respect to $t \in [0, t_0]$, i.e.*

$$T_t = \lim_{n\to\infty} [F(t/n)]^n. \tag{1}$$

A family of operators $(F(t))_{t\geq 0}$ is called *Chernoff equivalent* to the semigroup $(T_t)_{t\geq 0}$ (denoted $F(t) \sim T_t$) if this family satisfies the assertions of the Chernoff theorem with respect to $(T_t)_{t\geq 0}$, i.e. $F(t) \sim T_t \quad \Rightarrow \quad T_t = \lim_{n\to\infty} \left[F(t/n)\right]^n$.

Example (Trotter formula). Let L_1, L_2, $L_1 + L_2$ be generators of C_0-semigroups e^{tL_1}, e^{tL_2} and $e^{t(L_1+L_2)}$ on a Banach space X respectively, let L_1 and L_2 do not commute. Then $e^{tL_1} \circ e^{tL_2} \neq e^{t(L_1+L_2)} \neq e^{tL_2} \circ e^{tL_1}$. But $e^{tL_1} \circ e^{tL_2} \sim e^{t(L_1+L_2)}$, $\quad e^{tL_2} \circ e^{tL_1} \sim e^{t(L_1+L_2)}$, and due to the Chernoff Theorem

$$e^{t(L_1+L_2)} = \lim_{n\to\infty} \left[e^{\frac{t}{n}L_1} \circ e^{\frac{t}{n}L_2}\right]^n = \lim_{n\to\infty} \left[e^{\frac{t}{n}L_2} \circ e^{\frac{t}{n}L_1}\right]^n.$$

Rem.: The Post–Widder Inversion formula also follows from the Chernoff theorem: $e^{tL} = \lim_{n\to\infty} \left(\mathrm{Id} - \frac{t}{n}L\right)^{-n} \equiv \lim_{n\to\infty} \left[\frac{n}{t} R\left(n/t, L\right)\right]^n$.

Definition. A Feynman formula is a representation of a solution of an initial (or initial-boundary) value problem for an evolution equation (or, equivalently, a representation of the semigroup solving the problem) by a limit of n-fold iterated integrals as $n \to \infty$.

In many important cases the operators $F(t)$ in the equality (1) are integral operators, i.e.

$$T_t = \lim_{n\to\infty} \left[F(t/n)\right]^n = \lim_{N(n)\to\infty} \underbrace{\int \cdots \int}_{N(n)\ times} \ldots\ldots$$

and, hence, we have a limit of iterated integrals, i.e. a Feynman formula, on the right hand side of the equality (1). That's why in all cases the identity (1) is called Feynman formula.

Advantages of Feynman formulae:

- Possible to consider evolutionary equations with wide variety of operators L in the r.h.s. (e.g., with $L = A + B + C$, with non-local operators); initial-boundary value problems; evolutionary equations on different geometric structures (Riemannian manifolds, graphs, infinite dimensional spaces, p-adic spaces e.t.c.).
- In many cases it is possible to get the representation of the semigroup by the limit of iterated integrals of ELEMENTARY FUNCTIONS ONLY!!! This is useful for direct computations and computer modeling of the evolution, simulation of stochastic processes, etc.
- Feynman formulae allow to obtain new Feynman–Kac formulae, to define some new functional integrals (Feynman path integrals), to investigate relations between different functional integrals, to calculate path integrals numerically.

Results: Feynman formulae for multiplicative and additive perturbations

Q — metric space, $X = C_b(Q)$ or $X = C_\infty(Q)$ with the supremum norm $\|f\|_\infty = \sup_{q\in Q} |f(q)|$.

Let $(T_t)_{t\geq 0}$ be a C_0-semigroup on X with generator $(L, \mathrm{Dom}(L))$. Let $A(\cdot) : Q \to [\underline{\alpha}, \overline{\alpha}] \subset (0, \infty)$ be continuous on Q. Define $\widetilde{L}$:

$$\widetilde{L}\varphi(q) = A(q)(L\varphi)(q), \quad \mathrm{Dom}(\widetilde{L}) = \mathrm{Dom}(L).$$

Then $\widetilde{L}$ is also the generator of a C_0-semigroup which we denote by $(\widetilde{T}_t)_{t\geq 0}$. We call the operator $\widetilde{L}$ *a multiplicative perturbation* of the generator L, and the semigroup $(\widetilde{T}_t)_{t\geq 0}$ *the semigroup with the multiplicatively perturbed (by the function $A(\cdot)$) generator.*

Theorem 2. *Let the family $(F(t))_{t\geq 0}$ of bounded linear operators on X be Chernoff equivalent to the semigroup $(T_t)_{t\geq 0}$. Consider a family of operators $(\widetilde{F}(t))_{t\geq 0}$ defined by the formula*

$$\widetilde{F}(t)\varphi(q) = (F(A(q)t)\varphi)(q). \tag{2}$$

Then the family $(\widetilde{F}(t))_{t\geq 0}$ acts in X and is Chernoff equivalent to the semigroup $(\widetilde{T}_t)_{t\geq 0}$ with the generator, multiplicatively perturbed by the function $A(\cdot)$. Therefore, the following Feynman formula is valid locally uniformly with respect to $t \geq 0$ in $\mathcal{L}(X)$:

$$\widetilde{T}_t = \lim_{n\to\infty} \left[\widetilde{F}(t/n)\right]^n.$$

Corollary. *Consider a Markov process $(X_t)_{t\geq 0}$ with the state space Q and transition density $P(t, q, dy)$. Let the corresponding semigroup $(T_t)_{t\geq 0}$,*

$$T_t\varphi(q) = \mathbb{E}^q\left[\varphi(X_t)\right] \equiv \int_Q \varphi(y) P(t, q, dy),$$

be a strongly continuous semigroup on a Banach space X, where $X = C_b(Q)$ or $X = C_\infty(Q)$. Then by Theorem 2 the family $(\widetilde{F}(t))_{t\geq 0}$ defined by the formula

$$\widetilde{F}_t\varphi(q) = \int_Q \varphi(y) P(A(q)t, q, dy),$$

is Chernoff equivalent to the semigroup $(\widetilde{T}_t)_{t\geq 0}$ with multiplicatively perturbed (by the function $A(\cdot)$) generator. Therefore, the following Feynman formula is valid for any $\varphi \in X$ and any $q_0 \in Q$

$$\widetilde{T}_t\varphi(q_0) =$$
$$= \lim_{n\to\infty} \int_{Q^n} \varphi(q_n) P(A(q_0)t/n, q_0, dq_1) P(A(q_1)t/n, q_1, dq_2) \cdots P(A(q_{n-1})t/n, q_{n-1}, dq_n).$$

Rem.: A process $\widetilde{\xi}_t$, whose semigroup is $e^{t\widetilde{L}}$, can be obtained from ξ_t by some random time change. Note that $\quad \widetilde{\xi}_t \neq \xi_{A(\cdot)t} \quad$ and $\quad P^{\widetilde{\xi}}(t, q, dy) \neq P^{\xi}(A(q)t, q, dy)$!

Theorem 3. *Let X be a Banach space with a norm $\|\cdot\|_X$. Let $(T_k(t))_{t\geq 0}$, $k = 1, \ldots, m$, be strongly continuous semigroups on X with generators $(L_k, D(L_k))$ respectively. Assume that $L = L_1 + \cdots + L_m$ with domain $D(L) = \cap_{k=1}^m D(L_k)$ is closable and that the closure is the generator of a strongly continuous semigroup $(T(t))_{t\geq 0}$ on X. Let $(F_k(t))_{t\geq 0}$, $k = 1, \ldots, m$, be families of operators in X which are Chernoff equivalent to the semigroups $(T_k(t))_{t\geq 0}$ respectively, i.e. for each $k \in \{1, \ldots, m\}$ we have $F_k(0) = \mathrm{Id}$, $\|F_k(t)\| \leq e^{a_k t}$ for some $a_k > 0$ and there is a set $D_k \subset D(L_k)$, which is a core for L_k, such that $\lim_{t\to 0} \left\| \frac{F_k(t)\varphi - \varphi}{t} - L_k\varphi \right\|_X = 0$ for each $\varphi \in D_k$. Assume that there exists a set $D \subset \cap_{k=1}^m D_k$ which is a core for L. Then the family $(F(t))_{t\geq 0}$, where $F(t) = F_1(t) \circ \cdots \circ F_m(t)$ is Chernoff equivalent to the semigroup $(T(t))_{t\geq 0}$ and, hence, the Feynman formula*

$$T_t = \lim_{n\to\infty} \left[F(t/n)\right]^n$$

is valid in the strong operator topology locally uniformly with respect to $t \geq 0$.

Example. Let $X = C_\infty(\mathbb{R}^d)$, $B(\cdot) \in C_b(\mathbb{R}^d, \mathbb{R}^d)$, $C(\cdot) \in C_b(\mathbb{R}^d)$ and

$$L := \frac{1}{2}\Delta + B\nabla + C.$$

Take

$$F_1(t) = e^{tC},$$
$$F_2(t): \quad F_2(t)\varphi(q) = \varphi(q + tB(q)) \quad \Rightarrow \quad F_2(t) \sim e^{tB\nabla},$$
$$F_3(t) = e^{\frac{t}{2}\Delta}: \quad e^{\frac{t}{2}\Delta}\varphi(q) = (2\pi t)^{(-d/2)} \int_{\mathbb{R}^d} e^{-\frac{|q-y|^2}{2t}} \varphi(y) dy.$$

Then $F(t) \equiv F_1(t) \circ F_2(t) \circ F_3(t)$:

$$F(t)\varphi(q) = \frac{\exp(tC(q))}{\sqrt{(2\pi t)^d}} \int_{\mathbb{R}^d} \exp\left(-\frac{|q + tB(q) - y|^2}{2t}\right) \varphi(y) dy \sim e^{t(\frac{1}{2}\Delta + B\nabla + C)}.$$

Hence we obtain the Feynman formula for the semigroup $e^{t(\frac{1}{2}\Delta+B\nabla+C)}$:

$$e^{t(\frac{1}{2}\Delta+B\nabla+C)}\varphi(q_0) = \lim_{n\to\infty} \left(\left[F(t/n)\right]^n \varphi\right)(q_0) =$$
$$= \lim_{n\to\infty} (2\pi t/n)^{(-dn/2)} \int_{\mathbb{R}^d} \cdots \int_{\mathbb{R}^d} e^{\frac{t}{n}\sum_{k=1}^n C(q_{k-1})} e^{-\sum_{k=1}^n \frac{|q_{k-1}+B(q_{k-1})t/n - q_k|^2}{2t/n}} \varphi(q_n) dq_1 \cdots dq_n =$$
$$= \lim_{n\to\infty} \int_{\mathbb{R}^d} \cdots \int_{\mathbb{R}^d} e^{\frac{t}{n}\sum_{k=1}^n C(q_{k-1})} e^{-\sum_{k=1}^n B(q_{k-1})\cdot(q_{k-1}-q_k)} \times$$
$$\times e^{-\frac{t}{2n}\sum_{k=1}^n |B(q_{k-1})|^2} p^{BM}_{t/n}(q_0 - q_1) \cdots p^{BM}_{t/n}(q_{n-1} - q_n) \varphi(q_n) dq_1 \cdots dq_n =$$
$$= \mathbb{E}^{q_0}\left[e^{\int_0^t C(\xi_s)ds} e^{\int_0^t B(\xi_s)\cdot d\xi_s} e^{-\frac{1}{2}\int_0^t |B(\xi_s)|^2 ds} \varphi(\xi_t) \right].$$

with $p^{BM}_t(x) = (2\pi t)^{(-d/2)} \exp\left\{ -\frac{|x|^2}{2t} \right\}$ and Brownian motion ξ_t.

Rem.: Previous result is generalized for $L : L\varphi(q) = \mathrm{tr}(A(q)\,\mathrm{Hess}\varphi(q)) + B(q) \cdot \nabla\varphi(q) + C(q)\varphi(q)$ with $A : \mathbb{R}^d \to L(\mathbb{R}^d)$ be continuous and for any $x \in \mathbb{R}^d$ the operator $A(x)$ is symmetric and positive. Then $F(t) \sim e^{tL}$ with

$$F(t)\varphi(q) = \frac{\exp(tC(q))}{\sqrt{\det A(q)(2\pi t)^d}} \int_{\mathbb{R}^d} \exp\left(\frac{-A^{-1}(q)(q - y + tB(q)) \cdot (q - y + tB(q))}{2t}\right) \varphi(y) dy.$$

Rem.: In the similar way we obtain a Feynman formula for the Schrödinger type equation: let $X = L_2(\mathbb{R}^d)$, $B \in \mathbb{R}^d$, $C(\cdot) \in C_b(\mathbb{R}^d)$ and $L = \frac{i}{2}\Delta - B\nabla - iC$. Then

$$F(t): \ F(t)\varphi(q) = \frac{\exp(-itC(q))}{\sqrt{(2\pi i t)^d}} \int_{\mathbb{R}^d} \exp\left(\frac{i|q + tB(q) - y|^2}{2t}\right) \varphi(y) dy \sim e^{tL} \quad \text{and}$$

$$e^{tL}\varphi(q_0) = \lim_{n\to\infty} \left[e^{-i\frac{t}{n}C} \circ e^{-\frac{t}{n}B\nabla} \circ e^{\frac{i}{2}\Delta}\right]^n \varphi(q_0) =$$
$$= \lim_{n\to\infty} (2\pi i t)^{(-dn/2)} \int_{\mathbb{R}^d} \ldots \int_{\mathbb{R}^d} e^{-i\frac{t}{n}\sum_{k=1}^n C(q_{k-1})} e^{i\sum_{k=1}^n \frac{|q_k - q_{k-1} + \frac{t}{n}B|^2}{2t/n}} \varphi(q_n) dq_1 \ldots dq_n =$$
$$= \int e^{-i\int_0^t [C(\xi(s)) - \frac{1}{2}|B|^2] ds} e^{i\int_0^t B\cdot d\xi(s)} \varphi(\xi(t)) \Phi^{q_0}(d\xi),$$

where Φ^{q_0} is the Feynman pseudomeasure on the set of paths (starting at q_0) in the configuration space of a quantum system.

Results II: Feynman formula for the Cauchy–Dirichlet problem

Let $G \subset \mathbb{R}^d$ be a bounded domain with a smooth boundary, $X = C_\infty(\mathbb{R}^d)$, $(L, \mathrm{Dom}(L))$ be a dif. operator generating a C_0-semigroup $(e^{tL})_{t\ge 0}$ on X. Let $F(t) \sim e^{tL}$ and $F'(0) = L$ on a core D for L. Consider a CD problem:

$$\frac{\partial f}{\partial t}(t,q) = Lf(t,q), \quad t>0,\ q \in G,$$
$$f(0,q) = f_0(q), \quad q \in G,$$
$$f(t,q) = 0, \quad q \in \partial G.$$

Consider a Banach space $Y = C_0(\overline{G}) := \{\varphi \in C(G),\ \lim\limits_{q\to\partial G}\varphi(q) = 0\}$, $\|f\|_Y = \sup_{x\in G}|f(x)|$.

Assume that $\exists$ a C_0-semigroup $(e^{tL_o})_{t\ge0}$ on Y, resolving the CD problem: $f(t,q) = e^{tL_o}f_0(q)$. Assume that the set $D_o := \mathrm{Dom}(L_o) \cap \left[C^1(\mathbb{R}^d)\cap D\right]\big|_G$ is a core for the generator L_o of the semigroup $(e^{tL_o})_{t\ge0}$.

Theorem 4. *Consider $s : (0,\infty) \to (0,\infty)$, $s \in C^\infty(0,\infty)$, $s \downdownarrows 0$ as $t \downarrow 0$ and $s(t) = o(t)$. Let $G_{s(t)} \subset G$: $G_{s(t)} = \{q \in G \,|\, \mathrm{dist}(q,\partial G) > s(t)\}$. Let $(\psi_{s(t)})_{t>0}$ be a family of functions: $\lim_{t\to t^*}\|\psi_{s(t)} - \psi_{s(t^*)}\|_\infty = 0$ for all $t^* > 0$ and $\psi_{s(t)} : \mathbb{R}^d \to [0,1]$,*

$$\psi_{s(t)}(q) = \begin{cases} 1, & \text{for } q \in G_{s(t)}, \\ 0, & \text{for } q \in \mathbb{R}^d \setminus G, \end{cases}.$$

Consider $(F_o(t))_{t\ge0}$ in Y: $F_o(t)\varphi(q) = \psi_{s(t)}(q)[F(t)\varphi](q)$, where $F(t) \sim e^{tL}$. Then we have $F_o(t) \sim e^{tL_o}$ and hence

$$e^{tL_o} = \lim_{n\to\infty}\left[F_o(t/n)\right]^n.$$

Example. Consider the CD problem with the operator L: $L\phi(q) = \mathrm{tr}(A(q)\,\mathrm{Hess}\phi(q)) + B(q)\cdot\nabla\phi(q) + C(q)\phi(q)$. Then, due to Theorem 4 and the previous Example, for all $q_0 \in G$

$$f(t,q_0) = e^{tL_o}f_0(q_0) = \mathbb{E}^{q_0}\left[\exp\left(\int_0^t C(\xi_\tau)d\tau\right)f_0(\xi_t)\,\Big|\, t<\tau_G\right] =$$

$$= \lim_{n\to\infty}\int_{G^n} f_0(q_n)\exp\left(-\sum_{k=1}^n A^{-1}(q_{k-1})B(q_{k-1})\cdot(q_{k-1}-q_k)\right)\exp\left(\frac{t}{n}\sum_{k=1}^n C(q_{k-1})\right)\times$$
$$\times\exp\left(-\frac{t}{2n}\sum_{k=1}^n A^{-1}(q_{k-1})B(q_{k-1})\cdot B(q_{k-1})\right)p_A\!\left(\frac{t}{n},q_0,q_1\right)\cdots p_A\!\left(\frac{t}{n},q_{n-1},q_n\right)dq_1\ldots dq_n$$

where $\mathbb{E}^{q_0}$ is the expectation of a (starting at $q_0 \in G$) diffusion process ξ_t with variable diffusion matrix $A(\cdot)$ and drift $B(\cdot)$ killed on ∂G, τ_G is the time of reaching ∂G and

$$p_A(t,x,y) := \frac{1}{\sqrt{\det A(x)(2\pi t)^d}}\exp\left(-\frac{A^{-1}(x)(x-y)\cdot(x-y)}{2t}\right).$$

Applications

Feynman formulae for evolution on a star graph. Consider a star graph Γ with vertex v and $n \in \mathbb{N}$ external edges $l_1,\ldots,l_n$. Let ρ be the metric on Γ induced by the isomorphism $l_k \cong [0,+\infty)$; $\Gamma^o = \Gamma\setminus\{v\} = \sqcup_{k=1}^n l_k^o$, $l_k^o \cong (0,+\infty)$; $\xi \in l_k^o \Rightarrow \xi = (k,x)$ where $x = \rho(\xi,v) > 0$; $\varphi_k(x) := \varphi(\xi)\big|_{\xi\in l_k^o}$. Let $\int_\Gamma \varphi(\xi)d\xi := \sum_{k=1}^n\int_0^\infty \varphi_k(x)dx$. Let $C_\infty(\Gamma)$ be the Banach space of continuous functions on Γ vanishing at infinity equipped with the sup-norm $\|\cdot\|_\infty$. Let $C^2_\infty(\Gamma) = \{\varphi \in C_\infty(\Gamma) : \varphi \in C^2_\infty(\Gamma^o), \varphi''$ extends to Γ as a function in $C_\infty(\Gamma)\}$. Let δ_v be the Dirac delta-measure concentrated at the vertex v. Let $\rho_v(\xi,\eta) := \rho(\xi,v) + \rho(v,\eta)$ for all $\xi,\eta \in \Gamma$. Let $1_k(\xi) = 1$ if $\xi \in l_k^o$, $1_k(\xi) = 0$ if $\xi \notin l_k^o$. Let $g(t,z) = (2\pi t)^{-1/2}\exp\left\{\frac{-z^2}{2t}\right\}$. Define $p(t,\xi,\eta) = \sum_{k=1}^n 1_k(\xi)1_k(\eta)g(t,\rho(\xi,\eta))$, $p_v(t,\xi,\eta) = \sum_{k=1}^n 1_k(\xi)1_k(\eta)g(t,\rho_v(\xi,\eta))$, $p^D(t,\xi,\eta) = p(t,\xi,\eta) - p_v(t,\xi,\eta)$.

Proposition 1 (cf. [4], [5]). Let a, c, $b_k \in [0,1]$, $k = 1,\ldots,n$, $a \ne 1$ and $a + c + \sum_{k=1}^n b_k = 1$. Consider an operator L_1 on $C_\infty(\Gamma)$ with $\mathrm{Dom}(L_1) = \{\varphi \in C^2_\infty(\Gamma) : a\varphi(v) + \frac{c}{2}\varphi''(v) = \sum_{k=1}^n b_k\varphi_k'(v)\}$ and $L_1\varphi = \frac{1}{2}\varphi''$ for all $\varphi \in \mathrm{Dom}(L_1)$. Then $(L_1,\mathrm{Dom}(L_1))$ is the generator of a Feller semigroup $(T^1_t)_{t\ge0}$, $T^1_t\varphi(\xi) = \int_\Gamma \varphi(\eta)P(t,\xi,d\eta)$ for each $\varphi \in C_\infty(\Gamma)$.

The transition kernel $P(t,\xi,d\eta)$ is given explicitly by the following formulae:
for the case $a + c \in (0,1)$ with $w_k = \frac{b_k}{1-a-c}$, $\beta = \frac{a}{1-a-c}$, $\gamma = \frac{c}{1-a-c}$ and

$$g_{\beta,\gamma}(t,z) = \frac{1}{\gamma^2}(2\pi t)^{-1/2}\int_0^t \frac{s+\gamma z}{(t-s)^{3/2}}\exp\left\{-\frac{(s+\gamma z)^2}{2\gamma^2(t-s)}\right\}e^{-\beta s/\gamma}ds,$$

$$P(t,\xi,d\eta) = \tag{3}$$
$$= p^D(t,\xi,\eta)d\eta + \sum_{k,m=1}^n 1_k(\xi)1_m(\eta)2w_m g_{\beta,\gamma}(t,\rho_v(\xi,\eta))d\eta + \gamma g_{\beta,\gamma}(t,\rho(\xi,v))\delta_v(d\eta);$$

for the case $a + c = 0$ with $w_k = b_k$

$$P(t,\xi,\eta) = p^d(t,\xi,\eta)d\eta + \sum_{k,m=1}^n 1_k(\xi)1_m(\eta)2w_m g(t,\rho_v(\xi,\eta))d\eta; \tag{4}$$

for the case $a + c = 1$ with $a = \frac{\beta}{1+\beta}$, $\quad c = \frac{1}{1+\beta}$

$$P(t,\xi,\eta) = p^d(t,\xi,\eta)d\eta - \left(\int_0^t e^{-\beta(t-s)}\frac{\rho(\xi,v)}{\sqrt{2\pi s^3}}\exp\left\{-\frac{\rho(\xi,v)^2}{2s}\right\}ds,\right)\delta_v(d\eta). \tag{5}$$

Rem.: The heat kernel $P(t,\xi,d\eta)$ in (3) is the transition kernel of the process of Brownian motion on Γ constructed by killing (after an exponential holding time with the rate β at the vertex) the Walsh process (the analogue of the reflected Brownian motion) with sticky vertex with stickness parameter γ (see [4], [5] for the detailed exposition).

Let $A(\cdot) \in C(\Gamma)$, let $\exists\, 0 < \underline{\alpha} \le \overline{\alpha} < \infty$ such that $\underline{\alpha} \le A(\xi) \le \overline{\alpha}$ for all $\xi \in \Gamma$. Let $B(\cdot) \in C_b(\Gamma)$, $B(v) = 0$. Let $C(\cdot) \in C_b(\Gamma)$. As before let a, c, $b_k \in [0,1]$, $k = 1,\ldots,n$ with $a \ne 1$ and $a + c + \sum_{k=1}^n b_k = 1$. Consider an operator L such that for all $\varphi \in \mathrm{Dom}(L)$

$$L\varphi(\xi) = A(\xi)\varphi''(\xi) + B(\xi)\varphi'(\xi) + C(\xi)\varphi(\xi),$$

$$\mathrm{Dom}(L) = \left\{\varphi \in C^2_\infty(\Gamma) : a\varphi(v) + \frac{c}{2}\varphi''(v) = \sum_{k=1}^n b_k\varphi_k'(v)\right\}.$$

The operator $(L,\mathrm{Dom}(L))$ is the generator of a C_0-semigroup $(T_t)_{t\ge0}$ on the space $C_\infty(\Gamma)$.

Theorem 5. *Consider a family $(F_1(t))_{t\ge0}$ on $C_\infty(\Gamma)$ defined by*

$$F_1(t)\varphi(\xi) \equiv \widetilde{T^1_t}\varphi(\xi) := \int_\Gamma \varphi(\eta)P(A(\xi)t,\xi,d\eta).$$

Consider a family $(F_2(t))_{t\ge0}$ on $C_\infty(\Gamma)$ defined by

$$F_2(t)\varphi(\xi) \equiv \varphi(\xi + tB(\xi)) := \begin{cases}\varphi_k(x + tB_k(x)), & \xi = (k,x),\ x + tB_k(x) > 0,\\ \varphi(v), & \xi = (k,x),\ x + tB_k(x) \le 0.\end{cases}$$

Consider a family $(F_3(t))_{t\ge0}$ with $F_3(t)\varphi(\xi) = e^{tC(\xi)}\varphi(\xi)$, i.e. $F_3(t) = e^{tC}$. Then by Theorem 3, Theorem 2 and Proposition 1 the family $(F(t))_{t\ge0}$ with

$$F(t) = F_3(t)\circ F_2(t)\circ F_1(t): \quad F(t)\varphi(\xi) = e^{tC(\xi)}\int_\Gamma \varphi(\eta)P(A(\xi + tB(\xi))t, \xi + tB(\xi), d\eta) \tag{6}$$

is Chernoff equivalent to the semigroup $(T_t)_{t\ge0}$ on the space $C_\infty(\Gamma)$ generated by $(L,\mathrm{Dom}(L))$.

Feynman formulae for evolution in a Riemannian manifold. Consider a compact m-dimensional Riemannian manifold K, $G \subset K$ a domain with a smooth boundary ∂G, $\overline{G} = G \cup \partial G$, $A(\cdot) \in C(K)$, $A(x) \in [\underline{\alpha},\overline{\alpha}] \subset (0,+\infty)$ for each $x \in K$, $B(\cdot) \in C^1(K,TK)$, $C(\cdot) \in C(K)$ and the Cauchy–Dirichlet problem

$$\begin{cases}\frac{\partial f}{\partial t}(t,x) = -A(x)(\Delta_K f)(t,x) + B(x)\cdot\nabla f(t,x) + C(x)f(t,x), & t\ge0, x\in G,\\ f(0,x) = f_0(x), & x \in G,\\ f(t,x) = 0, & t \ge 0, x \in \partial G,\end{cases} \tag{7}$$

where $\Delta_K = -\mathrm{tr}\nabla^2$ — Laplace–Beltrami operator. Let vol_K be the Borel measure of Riemannian volume on K and ρ be the Riemannian metric in K. Due to the Nash theorem K is a smooth m-dimensional manifold isometrically embedded into a Euclidean space $\mathbb{R}^N$ and $\Phi : K \to \mathbb{R}^N$ is a smooth embedding. Define the following "pseudo-gaussian" functions for $t > 0$, $x, z \in K$:

$$g^E(t,x,z) = \frac{1}{(2\pi t)^{m/2}}e^{-\frac{\|\Phi(x)-\Phi(z)\|^2_{\mathbb{R}^N}}{2t}}, \qquad g^I(t,x,z) = \frac{1}{(2\pi t)^{m/2}}e^{-\frac{\rho(x,z)^2}{2t}},$$
$$p^E(t,x,z) = \frac{g^E(t,x,z)}{\int_K g^E(t,x,z)\mathrm{vol}_K(dz)}, \qquad p^I(t,x,z) = \frac{g^I(t,x,z)}{\int_K g^I(t,x,z)\mathrm{vol}_K(dz)}.$$

Let $\mathrm{scal}(x) \equiv \mathrm{tr}\,\mathrm{Ricci}(x)$ be a scalar curvature of K at the point $x \in K$. Let $r^2(x)$ be the square of the norm of the vector-valued mean curvature of K at the point x. Let $\mathrm{scal}(\cdot)$, $r^2(\cdot) \in C(K)$.

Proposition 2 (cf. [6], [1]). The following operator families are Chernoff equivalent to the semigroup $e^{-\frac{t}{2}\Delta_K}$ on the space $C(K)$:
$(F^E_p(t))_{t\ge0}: \quad (F^E_p(t)f)(x) = \int_K f(z)p^E(t,x,z)\mathrm{vol}_K(dz),$
$(F^E_g(t))_{t\ge0}: \quad (F^E_g(t)f)(x) = \int_K e^{\frac{t}{4}\mathrm{scal}(x)}e^{-\frac{t}{8}r^2(x)}f(z)g^E(t,x,z)\mathrm{vol}_K(dz),$
$(F^I_p(t))_{t\ge0}: \quad (F^I_p(t)f)(x) = \int_K f(z)p^I(t,x,z)\mathrm{vol}_K(dz),$
$(F^I_g(t))_{t\ge0}: \quad (F^I_g(t)f)(x) = \int_K e^{\frac{t}{6}\mathrm{scal}(x)}f(z)g^I(t,x,z)\mathrm{vol}_K(dz).$

Theorem 6. *Consider families $(\widetilde{F^E_p}(t))_{t\ge0}$, $(\widetilde{F^E_g}(t))_{t\ge0}$, $(\widetilde{F^I_p}(t))_{t\ge0}$, $(\widetilde{F^I_g}(t))_{t\ge0}$ obtained from families of Proposition 2 as in Theorem 2. Consider on $C(K)$ a family $(F_2(t))_{t\ge0}$: $(F_2(t)\varphi)(x) = \varphi(\gamma^x(t))$, where $\gamma^x(\cdot)$ is a geodesic: $\gamma^x(0) = x$ and $\dot\gamma^x(0) = A(x)$. Consider on $C(K)$ a family $(F_3(t))_{t\ge0}$ with $F_3(t)\varphi(x) = e^{tC(x)}\varphi(x)$. Let $(\psi_{s(t)})_{t\ge0}$ be constructed as in Theorem 4. Then by Theorem 2, Theorem 3, Theorem 4 and Proposition 2 the following families are Chernoff equivalent to the semigroup $(T_t)_{t\ge0}$ on $C_0(\overline{G})$, resolving the CD problem (7):*
$(S^E_p(t))_{t\ge0}: \quad S^E_p(t)\varphi(x) = \psi_{\varepsilon(t)}(x)\left[\circ F_3(t)\circ F_2(t)\circ \widetilde{F^E_p}(t)\,\varphi\right](x)$
$(S^E_g(t))_{t\ge0}: \quad S^E_g(t)\varphi(x) = \psi_{\varepsilon(t)}(x)\left[\circ F_3(t)\circ F_2(t)\circ \widetilde{F^E_g}(t)\,\varphi\right](x)$
$(S^I_p(t))_{t\ge0}: \quad S^I_p(t)\varphi(x) = \psi_{\varepsilon(t)}(x)\left[\circ F_3(t)\circ F_2(t)\circ \widetilde{F^I_p}(t)\,\varphi\right](x)$
$(S^I_g(t))_{t\ge0}: \quad S^I_g(t)\varphi(x) = \psi_{\varepsilon(t)}(x)\left[\circ F_3(t)\circ F_2(t)\circ \widetilde{F^I_g}(t)\,\varphi\right](x)$

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