Let \(X, \| \|_X \) be a Banach space, \( L(X) \) the space of all continuous linear operators on \( X \) equipped with the operator norm \( \| \cdot \| \), and \( D \) be the identity operator in \( X \). If \( Dom(L) \subseteq X \) is a linear subspace and \( L \), \( Dom(L) \) is a linear operator, then \( Dom(L) \) is a domain of \( L \). A one-parameter family \( \{ T(t) \}_{t \geq 0} \) of bounded linear operators \( T_{t} : X \to X \) is called a \( C_0 \)-semigroup, if \( T_{0} := Id \), \( T_{t+s} = T_{t}T_{s} \) for all \( t, s \geq 0 \) and \( \lim_{t \to \infty} \| T(t) - I \| \leq \| \). In probabilistic terms, \( L \) is the generator of a Markov process \(( Q_{t} , \psi_{0} ) \) with initial value \( \psi_{0} \) in \( X \) and \( \psi_{0} (x) \) given by \( \psi_{0} = T_{0} \psi_{0} \). Therefore, the following Feynman formula is valid for any \( x \in X \) and \( t > 0 \).

\[
T_{t}(\psi_{0}) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{1}{2t} \| x \|_{X}^{2}} \int_{X} e^{i(x,\xi)} \phi(\xi) d\xi.
\]

Corollary. Consider a Markov process \(( X_{\infty} , \psi_{0} ) \) with the state space \( Q \) and transition density \( F(\theta , \phi) \). Let the corresponding semigroup \(( T_{t} )_{t \geq 0} \),

\[
T_{t}(\psi_{0}) = \int_{Q} \phi(\theta) F(\theta , \phi_{0}) \psi(\theta) d\theta.
\]

Prove that the equality holds in the sense of distributions.

\[
\phi(\theta) = \int_{Q} \psi(\theta) F(\theta , \phi_{0}) d\theta.
\]

With this corollary, we can define the \( \Phi \)-semigroup in the configuration space of a quantum system.
Let $\mathbb{G} \subseteq \mathbb{R}^d$ be a bounded domain with a smooth boundary, $X = C_0(\mathbb{G})$ (the $L_{0}(\mathbb{G})$ be a differential operator generating a $C_{0}$-semigroup $(e^{\lambda t})_{t \geq 0}$ on $X$. Let $F(t) = e^{\lambda t}$ and $\phi(t) = L$. Consider a CD problem: $\frac{\partial F(t)}{\partial t} = LF(t)$, $t > 0$, $x \in \mathbb{G}$, $F(t, x) = \phi(t)$, $t = 0$, $x \in \mathbb{G}$.

Consider a Banach space $Y = C_{0}^{\infty}(\mathbb{G})$, $\rho_{G}(\phi(t)) = 0$, $\rho_{G}^{G}(f(x)) = 0$. Assume that $\exists \lambda > 0$, $\phi(t) = e^{\lambda t}f(t)$, $t > 0$, $x \in \mathbb{G}$. Consider the set $D = L_{0}\{\mathbb{G}\} \cap C_{0}^{\infty}(\mathbb{G}/\partial \mathbb{G})$, $\{\mathbb{G}\}$ be a generator for the semigroup $(e^{\lambda t})_{t \geq 0}$.

Theorem 4. Consider $\phi(t)$, $0 \leq t \leq \infty$, $\mathbb{G} \subset \mathbb{R}^d$, $\phi(t) \equiv e^{\lambda t}$ for all $t \geq 0$. Let $\mathcal{G}_{\mathbb{G}}(\phi(t)) = \{\mathbb{G}, \phi(t) \in \mathbb{G}/\partial \mathbb{G}\}$ be a family of functions: $\lim_{t \to \infty} \phi(t) = \lim_{t \to \infty} \phi(t) = 0$ for all $t > 0$, $x \in \mathbb{G}/\partial \mathbb{G}$. $\mathbb{G} \subseteq \mathbb{R}^d$. Consider $(F(t))_{t \geq 0}$ in $Y$: $F(t) = e^{\lambda t}f(t)
$. $F(t) = e^{\lambda t}f(t)$.

Then have $F(t) = e^{\lambda t}f(t)$.

Let $\lambda \in \mathbb{R}$ and hence.

\[
\lambda = \lim_{t \to 0} \frac{F(t) - F(0)}{t}.
\]

Applications

Feynman formulae for evolution on a star graph. Consider a star graph $T$ with vertices $v$ and $x \in \mathbb{N}$ external edges $y_{1}, \ldots, y_{N}$. Let $\rho$ be the metric on $T$ induced by the isomorphism $h : [0, \infty) \to \mathbb{G}$. Let $\Gamma = \mathbb{G}/\partial \mathbb{G}$, $\mathcal{G}_{\mathbb{G}}(T) = \{v, \mathbb{G} \in \mathbb{G}/\partial \mathbb{G}\}$. $\mathbb{G} \subseteq \mathbb{R}^d$ is a family function $\phi(t) = e^{\lambda t}$ for all $t \geq 0$. $x \in \mathbb{G}/\partial \mathbb{G}$.

Let $\phi(t) = e^{\lambda t}f(t)$, $t \geq 0$, $x \in \mathbb{G}/\partial \mathbb{G}$. $\mathbb{G} \subseteq \mathbb{R}^d$. Consider $(F(t))_{t \geq 0}$ in $Y$: $F(t) = e^{\lambda t}f(t)
$. $F(t) = e^{\lambda t}f(t)$.

Then have $F(t) = e^{\lambda t}f(t)$.

Let $\lambda \in \mathbb{R}$ and hence.

\[
\lambda = \lim_{t \to 0} \frac{F(t) - F(0)}{t}.
\]

References