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Some new results concerning the random difference equation.

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Random difference equation. We consider the Markov chain

$$X_0 = 0, \qquad X_n = A_n X_{n-1} + B_n = (A_n, B_n) \circ X_{n-1},$$

where $(A_n, B_n) \in \mathbb{R}^+ \times \mathbb{R}$ is an i.i.d. sequence.

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Existence of a stationary measure. Assume $\mathbb{E} \log A < 0$ and $\mathbb{E} \log^+ |B| < \infty$. Define the 'backward process'

$$\begin{aligned} R_n &= (A_1, B_1) \circ ... \circ (A_n, B_n) \circ X_0 \\ &= B_1 + A_1 B_2 + A_1 A_2 B_3 + ... + A_1..A_{n-1} B_n = \sum_{k=1}^{n} \Pi_{k-1} B_k \end{aligned}$$

where $\Pi_k = A_1..A_k$.

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where $\Pi_k = A_1 .. A_k$. Then R_n converges a.s. to

$$R = \sum_{k=1}^{\infty} \Pi_{k-1} B_k.$$

Since $X_n =_d R_n$, the process X_n converges in distribution to R and

$$R =_d AR + B, \qquad (A, B) \perp R$$

Then ν - the law of R, is the stationary measure of $\{X_n\}$.

If $\mathbb{E} \log A < 0$, then $R = \sum_{k=1}^{\infty} \Pi_{k-1} B_k$ is a unique solution of

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Theorem (Kesten 73, Grincevicius 75, Goldie 91)
If $\mathbb{E}A^\alpha = 1$ for some $\alpha > 0$, $\mathbb{E}[A^\alpha |\log A|] < \infty$ and ..., then

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We used this idea to study in more details large values of R. Our results:

- a formula for C_+
- ruin probability for R_n
- large deviations and ruin probability for $S_n = X_1 + .. + X_n$.

We want to study large values of $R = \sum_k \Pi_{k-1} B_k$.
R is larger than t mainly because $\Pi_n > \epsilon t$. Thus we are led to study large deviations of trajectories of Π_n. Our main tool is the **Bahadur-Rao theorem**.

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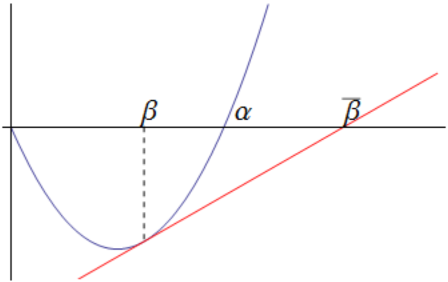

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"Proof": We change the measure $\mu(da) \mapsto \frac{1}{e^{\Lambda(\beta)}} a^\beta \mu(da)$ (we use $\mathbb{E}\big[\frac{A^\beta}{e^{\Lambda(\beta)}}\big] = 1$)

$$\begin{aligned} \mathbb{P}[A_1..A_n \approx t] &\approx \mathbb{E}\Big[\mathbf{1}_{\{A_1..A_n \approx t\}} \frac{(A_1..A_n)^\beta}{t^\beta}\Big] = t^{-\beta} e^{\Lambda(\beta) n} \mathbb{P}_\beta[A_1..A_n \approx t] \\ &\approx \frac{1}{t^{\overline{\beta}}} \mathbb{P}_\beta\Big[\sum_1^n \log A_k - n\Lambda'(\beta) \approx C\Big] \approx \frac{C}{\sqrt{n} t^{\overline{\beta}}} \end{aligned}$$

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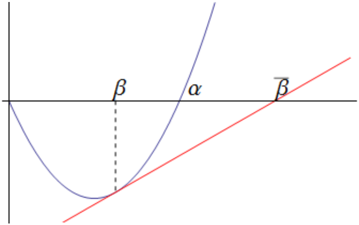

Notice that $\mathbb{P}[A_1 ... A_n > t]$ is the largest when $n = \frac{\log t}{\Lambda'(\alpha)}$. Therefore $R_n = \sum_{k=1}^n \Pi_{k-1} B_k$ exceeds t for $n \sim \frac{\log t}{\Lambda'(\alpha)}$.

A formula for C_+. Recall

$$X_n = A_n X_{n-1} + B_n, \qquad \mathbb{P}[R > t] \sim C_+ t^{-\alpha}.$$

Goldie found the following formula for C_+:

$$C_+ = \frac{1}{\Lambda'(\alpha)} \int_0^\infty \big(\mathbb{P}[AR + B > t] - \mathbb{P}[AR > t]\big) t^{\alpha - 1} dt.$$

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Theorem (E. Damek, J. Zienkiewicz, D.B.) Assume $B \geq 0$. Under assumptions of Kesten's theorem we have

$$C_+ = \frac{\alpha}{\Lambda'(\alpha)} \lim_{n\to\infty} \frac{\mathbb{E}X_n^\alpha}{n} = \frac{\alpha}{\Lambda'(\alpha)} \lim_{n\to\infty} \frac{\mathbb{E}R_n^\alpha}{n}$$

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If $B = 1$ then the corresponding formula appears in the paper of Bartkiewicz, Jakubowski, Mikosch, Winterberger (PTRF, 2011).

Sketch of the proof. $C_+ = \frac{\alpha}{\Lambda'(\alpha)} \lim_{n\to\infty} \frac{\mathbb{E}R_n^\alpha}{n}$, $R_n = \sum_{k=1}^n \Pi_{k-1} B_k$.

Let ν be the law of R. Take large M such that for $x > e^M$:
$\nu(dx) \sim \frac{C_+}{\alpha} \, \frac{dt}{t^{\alpha+1}}$. We write

$$\mathbb{E}R_L^\alpha = \mathbb{E}\big[R_L^\alpha \mathbf{1}_{\{R_L < e^M\}}\big] + \sum_{m \ge M} \mathbb{E}\big[R_L^\alpha \mathbf{1}_{\{e^m \le R_L < e^{m+1}\}}\big].$$

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Given large L we divide the sum into 3 parts

- $I = \left\{m : L < \frac{m}{\Lambda'(\alpha)} - m^{\frac{1}{2}+\delta}\right\}$
- $II = \left\{m : \frac{m}{\Lambda'(\alpha)} - m^{\frac{1}{2}+\delta} < L < \frac{m}{\Lambda'(\alpha)} + m^{\frac{1}{2}+\delta}\right\}$
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The number of elements of II is of the order $CL^{\frac{1}{2}+\delta}$. Then

$$\frac{\sum_{m\in II} \mathbb{E}[R_L^\alpha \mathbf{1}_{\{e^m\leq R_L<e^{m+1}\}}]}{L} \leq \frac{\sum_{m\in II} e^{\alpha m}\mathbb{P}[R > e^m]}{L} \leq \frac{C|II|}{L} \to 0$$

Finally we have to prove that

$$\frac{1}{L}\sum_{m\in III} R_L^{\alpha}\mathbf{1}_{\{e^m\le R_L<e^{m+1}\}} \to C_+,$$

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Notice that if $m \in III$, then on the set $\{e^m \le R_L < e^{m+1}\}$, R is very close to R_L. Moreover $|III| \sim L\Lambda'(\alpha)$. Hence

$$\begin{aligned}
\frac{1}{L}\sum_{m\in III}\mathbb{E}[R_L^\alpha \mathbf{1}_{\{e^m\le R_L<e^{m+1}\}}] &\sim \frac{1}{L}\sum_{m\in III}\mathbb{E}[R^\alpha \mathbf{1}_{\{e^m\le R<e^{m+1}\}}] \\
&= \frac{1}{L}\sum_{m\in III}\int_{\{e^m\le x<e^{m+1}\}} x^\alpha \nu(dx) \\
&\sim \frac{1}{L}\sum_{m\in III}\int_{\{e^m\le x<e^{m+1}\}} \frac{C_+}{\alpha}\frac{x^\alpha}{x^{\alpha+1}}dx \\
&\to \frac{\Lambda'(\alpha)}{\alpha}\cdot C_+
\end{aligned}$$

RUIN PROBABILITY

Recall: $R_n = \sum_{k=1}^{n} \Pi_{k-1} B_k$. Let $R_n^* = \max_{k \leq n} R_k$

Let $T = \inf\{n : \; R_n^* > t\}$. Our aim is to describe the law of T.

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Denote $n_0 = \frac{\log t}{\Lambda'(\alpha)}$.

- Take $n = \frac{\log t}{\Lambda'(\beta)}$ for $\beta > \alpha$ ($n < n_0$) then

$$\mathbb{P}[R_n^* > t] \sim \frac{C_\beta}{\sqrt{\log t}\ t^{\overline{\beta}}} \qquad \text{and} \qquad \mathbb{P}[R_{n-D\log n}^* > t] = o\left(\frac{C_\beta}{\sqrt{\log t}\ t^{\overline{\beta}}}\right)$$

- For n_0:

$$\mathbb{P}[R_{n_0}^* > t] \sim \frac{C_+}{2t^\alpha} \qquad \text{and} \qquad \mathbb{P}[R_{n_0 - M\sqrt{n_0}}^* > t] \sim \frac{1}{e^M}\frac{C}{t^\alpha}$$

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Sketch of the proof: $\mathbb{P}[R_{n_0} > t] \sim \frac{C_+}{2t^\alpha}$, $t = e^{n_0 \Lambda'(\alpha)}$

Fix large M and define $n_1 = n_0 - M\sqrt{n_0}$. We have

$$R_{n_0} = \sum_{k=1}^{n_0} \Pi_{k-1} B_k = R_{n_1} + \Pi_{n_1} \cdot \sum_{k=n_1}^{n_0} A_{n_1+1} \ldots A_{k-1} B_k = R_{n_1} + \Pi_{n_1} R'_{M\sqrt{n_0}}$$

Notice $R'_{M\sqrt{n_0}} =_d R_{M\sqrt{n}}$.

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Notice $R'_{M\sqrt{n_0}} =_d R_{M\sqrt{n}}$. It is sufficient to study

$$\begin{aligned}
\mathbb{P}\big[\Pi_{n_1} R'_{M\sqrt{n_0}} > t\big] &= \int_{\mathbb{R}^+} \mathbb{P}\big[\Pi_{n_1} > \frac{t}{x}\big] \mathbb{P}\big[R_{M\sqrt{n_0}} \in dx\big] \\
&= \int_{\mathbb{R}^+} \mathbb{P}\Big[\Pi_{n_1} > e^{\Lambda'(\alpha) n_1 + \Lambda'(\alpha) M\sqrt{n_0} - \log x}\Big] \mathbb{P}\big[R_{M\sqrt{n_0}} \in dx\big] \\
&\sim \frac{C}{\sqrt{n_0}} \int_{\mathbb{R}^+} \frac{x^\alpha}{t^\alpha} e^{-\frac{1}{n_0}(M\sqrt{n_0}\Lambda'(\alpha) - \log x)^2} \mathbb{P}\big[R_{M\sqrt{n_0}} \in dx\big] \\
&\sim \frac{C}{\sqrt{n_0}} \int_{\mathbb{R}^+} \frac{x^\alpha}{t^\alpha} e^{-\frac{1}{n_0}(M\sqrt{n_0}\Lambda'(\alpha) - \log x)^2} \mathbb{P}\big[R \in dx\big] \\
&\sim \frac{C}{t^\alpha} \cdot \frac{1}{\sqrt{n_0}} \int_{\mathbb{R}^+} x^\alpha e^{-\frac{1}{n_0}(M\sqrt{n_0}\Lambda'(\alpha) - \log x)^2} \frac{C_+ dx}{x^{\alpha+1}} \\
&= \frac{CC_+}{t^\alpha} \cdot \frac{1}{\sqrt{n_0}} \int_{\mathbb{R}} e^{-\frac{1}{n_0}(M\sqrt{n_0}\Lambda'(\alpha) - y)^2} dy = \frac{CC_+}{t^\alpha} \int_{\mathbb{R}} e^{-y^2} dy
\end{aligned}$$

Theorem (J.Collamore, E. Damek, J. Zienkiewicz, D.B.)
Denote $n_0 = \frac{\log t}{\Lambda'(\alpha)}$.

- Take $n = \frac{\log t}{\Lambda'(\beta)}$ for $\beta > \alpha$ ($n < n_0$) then

$$\mathbb{P}[R_n^* > t] \sim \frac{C_\beta}{\sqrt{\log t}\ t^{\overline{\beta}}} \qquad \text{and} \qquad \mathbb{P}[R_{n-D\log n}^* > t] = o\left(\frac{C_\beta}{\sqrt{\log t}\ t^{\overline{\beta}}}\right)$$

- For n_0:

$$\mathbb{P}[R_{n_0}^* > t] \sim \frac{C_+}{2t^\alpha} \qquad \text{and} \qquad \mathbb{P}[R_{n_0 - M\sqrt{n_0}}^* > t] \sim \frac{1}{e^M}\frac{C}{t^\alpha}$$

We still don't understand the case when $n > n_0$, $n = \frac{\log t}{\Lambda'(\beta)}$ for $\beta < \alpha$.

Natural conjecture: $\mathbb{P}\big[R_n < t, R > t\big] \sim \frac{1}{\sqrt{\log t}\ t^{\overline{\beta}}}$ is wrong.
We constructed an example where $\mathbb{P}\big[R_n < t, R_{n+1} > t\big] > \frac{C}{t^\gamma}$ for some $\gamma < \overline{\beta}$

LARGE DEVIATIONS and RUIN PROBABILITY for sums $S_n = X_1 + .. + X_n$, where $X_n = A_n X_{n-1} + B_n$.

Theorem (E. Damek, T. Mikosch, J. Zienkiewicz, D.B; AoP 2013)

$$\lim_{n\to\infty} \sup_{x\in[a_n,b_n]} \frac{\mathbb{P}[S_n - \mathbb{E}S_n > x]}{n\mathbb{P}[R > x]} \to C_1 > 0$$

If $\alpha > 1$ and $\mu > 0$, then

$$\lim_{n\to\infty} \frac{\mathbb{P}[\sup_n (S_n - \mathbb{E}S_n - n\mu) > x]}{x\mathbb{P}[R > x]} \to C_2 > 0$$