



Uniwersytet  
Wrocławski

# Some new results concerning the random difference equation.

Dariusz Buraczewski

joint with Jeffrey Collamore, Thomas Mikosch (Copenhagen)

Ewa Damek, Jacek Zienkiewicz (Wrocław)

Wrocław, 17 VII 2013

**Random difference equation.** We consider the Markov chain

$$X_0 = 0, \quad X_n = A_n X_{n-1} + B_n = (A_n, B_n) \circ X_{n-1},$$

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**Existence of a stationary measure.** Assume  $\mathbb{E} \log A < 0$  and  $\mathbb{E} \log^+ |B| < \infty$ . Define the 'backward process'

$$\begin{aligned} R_n &= (A_1, B_1) \circ \dots \circ (A_n, B_n) \circ X_0 \\ &= B_1 + A_1 B_2 + A_1 A_2 B_3 + \dots + A_1 \dots A_{n-1} B_n = \sum_{k=1}^n \Pi_{k-1} B_k \end{aligned}$$

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where  $\Pi_k = A_1 \dots A_k$ . Then  $R_n$  converges a.s. to

$$R = \sum_{k=1}^{\infty} \Pi_{k-1} B_k.$$

Since  $X_n =_d R_n$ , the process  $X_n$  converges in distribution to  $R$  and

$$R =_d AR + B, \quad (A, B) \perp R$$

Then  $\nu$  - the law of  $R$ , is the stationary measure of  $\{X_n\}$ .

If  $\mathbb{E} \log A < 0$ , then  $R = \sum_{k=1}^{\infty} \Pi_{k-1} B_k$  is a unique solution of

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**Theorem (Kesten 73, Grincevicius 75, Goldie 91)**

If  $\mathbb{E}A^\alpha = 1$  for some  $\alpha > 0$ ,  $\mathbb{E}[A^\alpha | \log A] < \infty$  and ..., then

$$\mathbb{P}[R > t] \sim C_+ t^{-\alpha}$$

in other words if  $\nu$  is the law of  $R$ , then

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We used this idea to study in more details large values of  $R$ . Our results:

- a formula for  $C_+$
- ruin probability for  $R_n$
- large deviations and ruin probability for  $S_n = X_1 + \dots + X_n$ .



We want to study large values of  $R = \sum_k \Pi_{k-1} B_k$ .

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**Bahadur-Rao Theorem:** Fix  $\beta > 0$  such that  $\Lambda'(\beta) > 0$ , let  $n = \frac{\log t}{\Lambda'(\beta)}$  ( $t = e^{n\Lambda'(\beta)}$ ) and  $\bar{\beta} = \beta - \frac{\Lambda(\beta)}{\Lambda'(\beta)}$ . Then

$$\mathbb{P}[A_1 \dots A_n > t] \sim \frac{C}{\sqrt{\log t} t^{\bar{\beta}}}$$

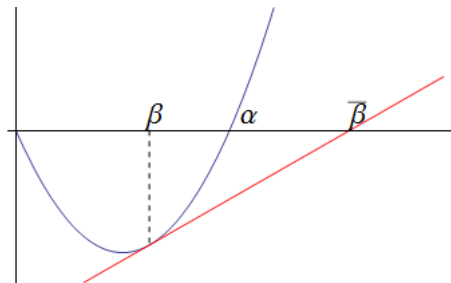
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"Proof": We change the measure  $\mu(da) \mapsto \frac{1}{e^{\Lambda(\beta)}} a^\beta \mu(da)$  (we use  $\mathbb{E} \left[ \frac{A^\beta}{e^{\Lambda(\beta)}} \right] = 1$ )

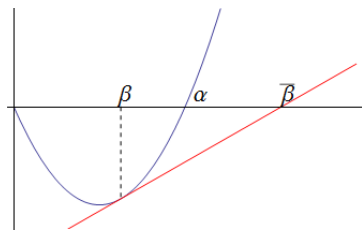
$$\begin{aligned} \mathbb{P}[A_1 \dots A_n \approx t] &\approx \mathbb{E} \left[ \mathbf{1}_{\{A_1 \dots A_n \approx t\}} \frac{(A_1 \dots A_n)^\beta}{t^\beta} \right] = t^{-\beta} e^{\Lambda(\beta)n} \mathbb{P}_\beta[A_1 \dots A_n \approx t] \\ &\approx \frac{1}{t^{\bar{\beta}}} \mathbb{P}_\beta \left[ \sum_1^n \log A_k - n\Lambda'(\beta) \approx C \right] \approx \frac{C}{\sqrt{nt} t^{\bar{\beta}}} \end{aligned}$$

We want to study large values of  $R = \sum_k \Pi_{k-1} B_k$ .  $R > t$  mainly because  $\Pi_n > \epsilon t$ . Thus we are led to study large deviations of trajectories of  $\Pi_n$ . Our main tool is the **Bahadur-Rao theorem**.

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Notice that  $\mathbb{P}[A_1 \dots A_n > t]$  is the largest when  $n = \frac{\log t}{\Lambda'(\alpha)}$ . Therefore  $R_n = \sum_{k=1}^n \Pi_{k-1} B_k$  exceeds  $t$  for  $n \sim \frac{\log t}{\Lambda'(\alpha)}$ .

**A formula for  $C_+$ .** Recall

$$X_n = A_n X_{n-1} + B_n, \quad \mathbb{P}[R > t] \sim C_+ t^{-\alpha}.$$

Goldie found the following formula for  $C_+$ :

$$C_+ = \frac{1}{\Lambda'(\alpha)} \int_0^\infty (\mathbb{P}[AR + B > t] - \mathbb{P}[AR > t]) t^{\alpha-1} dt.$$

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Recently Enriquez, Sabot, Zindy (PTRF 2009), Collamore, Vidyashankar (2012) found formulas for  $C_+$  in terms of other processes or stopping time related to behavior of  $\Pi_n$  (under additional strong assumptions e.g.  $B = 1$  or continuity of the distribution).



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**Theorem (E. Damek, J. Zienkiewicz, D.B.)** Assume  $B \geq 0$ . Under assumptions of Kesten's theorem we have

$$C_+ = \frac{\alpha}{\Lambda'(\alpha)} \lim_{n \rightarrow \infty} \frac{\mathbb{E}X_n^\alpha}{n} = \frac{\alpha}{\Lambda'(\alpha)} \lim_{n \rightarrow \infty} \frac{\mathbb{E}R_n^\alpha}{n}$$

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If  $B = 1$  then the corresponding formula appears in the paper of Bartkiewicz, Jakubowski, Mikosch, Winterberger (PTRF, 2011).

**Sketch of the proof.**  $C_+ = \frac{\alpha}{\Gamma'(\alpha)} \lim_{n \rightarrow \infty} \frac{\mathbb{E}R_n^\alpha}{n}$ ,  $R_n = \sum_{k=1}^n \prod_{k-1} B_k$ .

Let  $\nu$  be the law of  $R$ . Take large  $M$  such that for  $x > e^M$ :

$\nu(dx) \sim \frac{C_+}{\alpha} \frac{dt}{t^{\alpha+1}}$ . We write

$$\mathbb{E}R_L^\alpha = \mathbb{E}[R_L^\alpha \mathbf{1}_{\{R_L < e^M\}}] + \sum_{m \geq M} \mathbb{E}[R_L^\alpha \mathbf{1}_{\{e^m \leq R_L < e^{m+1}\}}].$$

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Given large  $L$  we divide the sum into 3 parts

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- $II = \{m : \frac{m}{\Lambda'(\alpha)} - m^{\frac{1}{2}+\delta} < L < \frac{m}{\Lambda'(\alpha)} + m^{\frac{1}{2}+\delta}\}$
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The number of elements of  $II$  is of the order  $CL^{\frac{1}{2}+\delta}$ . Then

$$\frac{\sum_{m \in II} \mathbb{E}[R_L^\alpha \mathbf{1}_{\{e^m \leq R_L < e^{m+1}\}}]}{L} \leq \frac{\sum_{m \in II} e^{\alpha m} \mathbb{P}[R > e^m]}{L} \leq \frac{C|III|}{L} \rightarrow 0$$

Finally we have to prove that

$$\frac{1}{L} \sum_{m \in III} R_L^\alpha \mathbf{1}_{\{e^m \leq R_L < e^{m+1}\}} \rightarrow C_+,$$

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Notice that if  $m \in III$ , then on the set  $\{e^m \leq R_L < e^{m+1}\}$ ,  $R$  is very close to  $R_L$ . Moreover  $|III| \sim L\Lambda'(\alpha)$ . Hence

$$\begin{aligned} \frac{1}{L} \sum_{m \in III} \mathbb{E}[R_L^\alpha \mathbf{1}_{\{e^m \leq R_L < e^{m+1}\}}] &\sim \frac{1}{L} \sum_{m \in III} \mathbb{E}[R^\alpha \mathbf{1}_{\{e^m \leq R < e^{m+1}\}}] \\ &= \frac{1}{L} \sum_{m \in III} \int_{\{e^m \leq x < e^{m+1}\}} x^\alpha \nu(dx) \\ &\sim \frac{1}{L} \sum_{m \in III} \int_{\{e^m \leq x < e^{m+1}\}} \frac{C_+}{\alpha} \frac{x^\alpha}{x^{\alpha+1}} dx \\ &\rightarrow \frac{\Lambda'(\alpha)}{\alpha} \cdot C_+ \end{aligned}$$



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- For  $n_0$ :

$$\mathbb{P}[R_{n_0}^* > t] \sim \frac{C_+}{2t^\alpha} \quad \text{and} \quad \mathbb{P}[R_{n_0-M\sqrt{n_0}}^* > t] \sim \frac{1}{e^M} \frac{C}{t^\alpha}$$

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We assume  $B \geq 0$ , then  $R_n^* = R_n$ .

**Sketch of the proof:**  $\mathbb{P}[R_{n_0} > t] \sim \frac{C_+}{2t^\alpha}$ ,  $t = e^{n_0 \Lambda'(\alpha)}$

Fix large  $M$  and define  $n_1 = n_0 - M\sqrt{n_0}$ . We have

$$R_{n_0} = \sum_{k=1}^{n_0} \Pi_{k-1} B_k = R_{n_1} + \Pi_{n_1} \cdot \sum_{k=n_1}^{n_0} A_{n_1+1} \dots A_{k-1} B_k = R_{n_1} + \Pi_{n_1} R'_{M\sqrt{n_0}}$$

Notice  $R'_{M\sqrt{n_0}} =_d R_{M\sqrt{n}}$ .

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Fix large  $M$  and define  $n_1 = n_0 - M\sqrt{n_0}$ . We have

$$R_{n_0} = \sum_{k=1}^{n_0} \Pi_{k-1} B_k = R_{n_1} + \Pi_{n_1} \cdot \sum_{k=n_1}^{n_0} A_{n_1+1} \dots A_{k-1} B_k = R_{n_1} + \Pi_{n_1} R'_{M\sqrt{n_0}}$$

Notice  $R'_{M\sqrt{n_0}} =_d R_{M\sqrt{n}}$ . It is sufficient to study

$$\begin{aligned} \mathbb{P}[\Pi_{n_1} R'_{M\sqrt{n_0}} > t] &= \int_{\mathbb{R}^+} \mathbb{P}[\Pi_{n_1} > \frac{t}{x}] \mathbb{P}[R_{M\sqrt{n_0}} \in dx] \\ &= \int_{\mathbb{R}^+} \mathbb{P}[\Pi_{n_1} > e^{\Lambda'(\alpha)n_1 + \Lambda'(\alpha)M\sqrt{n_0} - \log x}] \mathbb{P}[R_{M\sqrt{n_0}} \in dx] \\ &\sim \frac{C}{\sqrt{n_0}} \int_{\mathbb{R}^+} \frac{x^\alpha}{t^\alpha} e^{-\frac{1}{n_0}(M\sqrt{n_0}\Lambda'(\alpha) - \log x)^2} \mathbb{P}[R_{M\sqrt{n_0}} \in dx] \\ &\sim \frac{C}{\sqrt{n_0}} \int_{\mathbb{R}^+} \frac{x^\alpha}{t^\alpha} e^{-\frac{1}{n_0}(M\sqrt{n_0}\Lambda'(\alpha) - \log x)^2} \mathbb{P}[R \in dx] \\ &\sim \frac{C}{t^\alpha} \cdot \frac{1}{\sqrt{n_0}} \int_{\mathbb{R}^+} x^\alpha e^{-\frac{1}{n_0}(M\sqrt{n_0}\Lambda'(\alpha) - \log x)^2} \frac{C_+ dx}{x^{\alpha+1}} \\ &= \frac{CC_+}{t^\alpha} \cdot \frac{1}{\sqrt{n_0}} \int_{\mathbb{R}} e^{-\frac{1}{n_0}(M\sqrt{n_0}\Lambda'(\alpha) - y)^2} dy = \frac{CC_+}{t^\alpha} \int_{\mathbb{R}} e^{-y^2} dy \end{aligned}$$

## Theorem (J.Collamore, E. Damek, J. Zienkiewicz, D.B.)

Denote  $n_0 = \frac{\log t}{\Lambda'(\alpha)}$ .

- Take  $n = \frac{\log t}{\Lambda'(\beta)}$  for  $\beta > \alpha$  ( $n < n_0$ ) then

$$\mathbb{P}[R_n^* > t] \sim \frac{C_\beta}{\sqrt{\log t} t^\beta} \quad \text{and} \quad \mathbb{P}[R_{n-D \log n}^* > t] = o\left(\frac{C_\beta}{\sqrt{\log t} t^\beta}\right)$$

- For  $n_0$ :

$$\mathbb{P}[R_{n_0}^* > t] \sim \frac{C_+}{2t^\alpha} \quad \text{and} \quad \mathbb{P}[R_{n_0 - M\sqrt{n_0}}^* > t] \sim \frac{1}{e^M} \frac{C}{t^\alpha}$$

We still don't understand the case when  $n > n_0$ ,  $n = \frac{\log t}{\Lambda'(\beta)}$  for  $\beta < \alpha$ .

Natural conjecture:  $\mathbb{P}[R_n < t, R > t] \sim \frac{1}{\sqrt{\log t} t^\beta}$  is wrong.

We constructed an example where  $\mathbb{P}[R_n < t, R_{n+1} > t] > \frac{C}{t^\gamma}$  for some  $\gamma < \bar{\beta}$

## LARGE DEVIATIONS and RUIN PROBABILITY for sums

$S_n = X_1 + \dots + X_n$ , where  $X_n = A_n X_{n-1} + B_n$ .

**Theorem (E. Damek, T. Mikosch, J. Zienkiewicz, D.B; AoP 2013)**

$$\lim_{n \rightarrow \infty} \sup_{x \in [a_n, b_n]} \frac{\mathbb{P}[S_n - \mathbb{E}S_n > x]}{n\mathbb{P}[R > x]} \rightarrow C_1 > 0$$

If  $\alpha > 1$  and  $\mu > 0$ , then

$$\lim_{n \rightarrow \infty} \frac{\mathbb{P}[\sup_n (S_n - \mathbb{E}S_n - n\mu) > x]}{x\mathbb{P}[R > x]} \rightarrow C_2 > 0$$