Superposition of COGARCH processes

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The COGARCH recalled

Let $(L_t)_{t\geq 0}$ be a Lévy process with characteristic triplet $(\gamma_L, \sigma_L^2, \nu_L)$ and define for $\eta > 0$ and $\varphi > 0$ the Lévy processes

$$S_t := [L, L]_t^{\mathsf{d}} = \sum_{0<s\leq t} (\Delta L_s)^2, \quad t \geq 0, \quad \text{and} \quad X_t^\varphi := \eta t - \sum_{0<s\leq t} \log(1 + \varphi \Delta S_s), \quad t \geq 0.$$

The **COGARCH (volatility) process** V^φ driven by the Lévy process L (or the subordinator S) with parameter φ and its corresponding **price process** or **integrated COGARCH process** G are defined as

$$V_t^\varphi = \mathsf{e}^{-X_t^\varphi}\left(V_0^\varphi + \beta \int_{(0,t]} \mathsf{e}^{X_s^\varphi}\, \mathsf{d}s\right), \quad \text{and} \quad G_t = \int_0^t \sqrt{V_{s-}^\varphi}\, \mathsf{d}L_s, \quad t \geq 0, \qquad (1)$$

where $\beta > 0$ is a constant and V_0^φ is a nonnegative random variable, independent of $(L_t)_{t\geq 0}$.

Some **important features** of the COGARCH process:

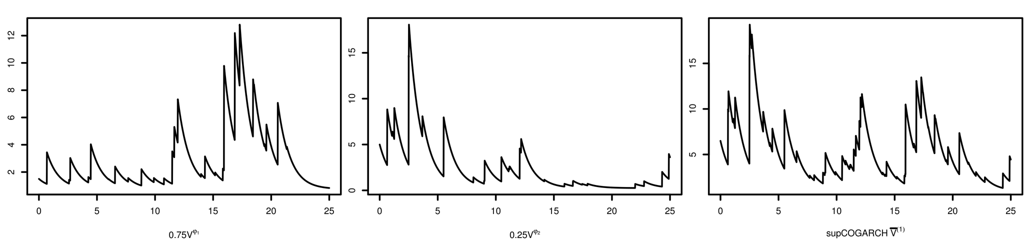
- The COGARCH volatility V^φ has a strictly stationary distribution if and only if $\int_{\mathbb{R}_+} \log(1 + \varphi y)\, \nu_S(\mathsf{d}y) = \int_{\mathbb{R}} \log(1 + \varphi y^2)\, \nu_L(\mathsf{d}y) < \eta$. We denote the set of all $\varphi > 0$ where a stationary distribution exists by $\Phi_L = (0, \varphi_{\max})$.
- The stationary distribution is uniquely determined by the law of $V_\infty^\varphi := \beta \int_{\mathbb{R}_+} \mathsf{e}^{-X_s^\varphi}\, \mathsf{d}s$.
- For $\kappa > 0$ constant, $\mathbb{E}[S_1^{\max\{\kappa,1\}}] < \infty$ and $\Psi(\kappa, \varphi) := \log \mathbb{E}\left[\mathsf{e}^{-\kappa X_1^\varphi}\right] < 0$ imply $\mathbb{E}[|V_0^\varphi|^\kappa] < \infty$.
- First and second moment of the stationary COGARCH are given by

$$\mathbb{E}[V_t^\varphi] = -\frac{\beta}{\Psi(1,\varphi)} = \frac{\beta}{\eta - \varphi \mathbb{E}[S_1]},$$
$$\mathsf{Cov}\,[V_t^\varphi, V_{t+h}^\varphi] = \mathsf{e}^{h\,\Psi(1,\varphi)} \mathsf{Var}\,[V_0^\varphi] = \mathsf{e}^{h\,\Psi(1,\varphi)} \beta^2 \left(\frac{2}{\Psi(1,\varphi)\Psi(2,\varphi)} - \frac{1}{\Psi(1,\varphi)^2}\right).$$

- Typically the stationary distribution has Pareto-like tails.
- The price process of a stationary COGARCH has stationary increments which are uncorrelated on disjoint intervals. The squared increments are, under some technical assumptions, positively correlated.
- Extending L to a two-sided Lévy process naturally leads to the two-sided COGARCH process $V_t^\varphi := \beta \int_{(-\infty,t]} \mathsf{e}^{-(X_t^\varphi - X_s^\varphi)}\, \mathsf{d}s$, $t \in \mathbb{R}$, which is always stationary.
- The COGARCH (V^φ, G) is a Markov process with respect to its natural filtration.

The supCOGARCH 1

Weighted sum of independent COGARCHes



Sample paths of two independent COGARCH processes and the resulting supCOGARCH 1 process.

The **supCOGARCH 1 volatility process** is defined as

$$\bar{V}_t^{(1)} = \int_{\mathbb{R}_+} V_t^\varphi\, \pi(\mathsf{d}\varphi) = \sum_{i=1}^\infty p_i V_t^{\varphi_i}, \quad t \geq 0,$$

where $\pi = \sum_{i=1}^\infty p_i \delta_{\varphi_i}$ for nonnegative weights $(p_i)_{i\in\mathbb{N}}$ with $\sum_{i=1}^\infty p_i = 1$. Each COGARCH V^φ is driven by an L^φ where $(L^\varphi)_{\varphi\in\mathbb{R}_+}$ are i.i.d. copies of a canonical Lévy process L.

To avoid degenerate cases we assume $\bar{V}_0^{(1)} < \infty$ a.s.

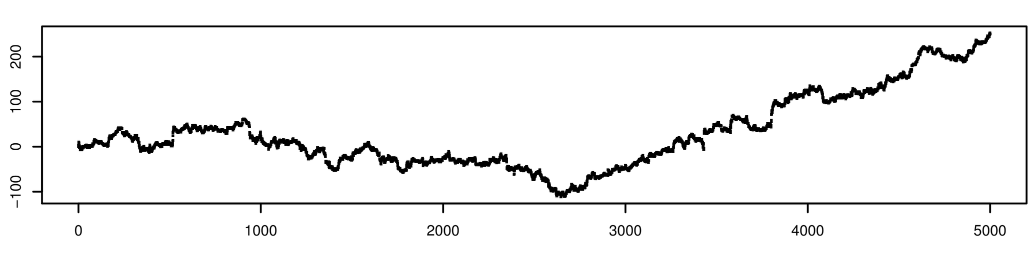
Main features:

- The jumps of the supCOGARCH 1 are given by $\Delta\bar{V}_t^{(1)} = \sum_{i=1}^\infty p_i \Delta V_t^{\varphi_i} = \sum_{i=1}^\infty p_i V_{t-}^{\varphi_i} \varphi_i \Delta S_t^{\varphi_i}, \quad t \geq 0,$ where a.s. only one summand is nonzero at each jump.
- A finite random variable $\bar{V}_0^{(1)}$ can be chosen such that $\bar{V}^{(1)}$ is strictly stationary if and only if $\pi(\Phi_L) = 1$.
- The stationary distribution is uniquely determined by the law of $\bar{V}_\infty^{(1)} := \int_{\Phi_L} V_\infty^\varphi\, \pi(\mathsf{d}\varphi) = \beta \sum_{i=1}^\infty p_i \int_{\mathbb{R}_+} \mathsf{e}^{-X_t^{\varphi_i}}\, \mathsf{d}t$ and it is self-decomposable.
- First and second moment of the stationary process are given by $\mathbb{E}[\bar{V}_t^{(1)}] = \int_{\mathbb{R}_+} \mathbb{E}[V_0^\varphi]\, \pi(\mathsf{d}\varphi) \leq \infty$, and $\mathsf{Cov}\,[\bar{V}_t^{(1)}, \bar{V}_{t+h}^{(1)}] = \sum_{i=1}^\infty p_i^2 \mathsf{Cov}\,[V_0^{\varphi_i}, V_h^{\varphi_i}] \leq \infty$.
- Typically the stationary distribution has Pareto-like tails.
- $\bar{V}^{(1)}$ is no Markov process with respect to its natural filtration.

The **integrated supGOGARCH 1 process** may be defined as

$$G_t^{(1)} := \int_{(0,t]} \sqrt{\bar{V}_{s-}^{(1)}}\, \mathsf{d}L_s^{\varphi_1}, \quad t \geq 0.$$

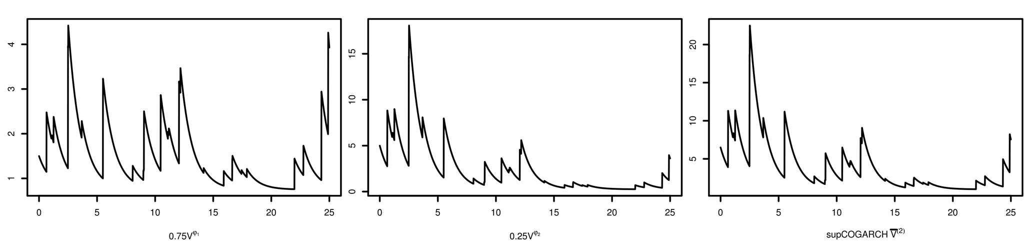
- This definition allows for jumps only in the volatility and not in the price process.
- We could also take any other function of $(L^{\varphi_i})_{i\in\mathbb{N}}$ as integrator and obtain various different price processes.
- If $(\bar{V}_t^{(1)})_{t\geq 0}$ is strictly stationary, then $(G_t^{(1)})_{t\geq 0}$ has stationary increments. Furthermore, its second-order structure is similar to that of the integrated COGARCH process.



Sample path of an integrated supCOGARCH 1 process.

The supCOGARCH 2

Weighted integral of dependent COGARCHes



Sample paths of two dependent COGARCH processes and the resulting supCOGARCH 2 process.

The **supCOGARCH 2 volatility process** is defined for $t \in \mathbb{R}$ as

$$\bar{V}_t^{(2)} := \int_{\Phi_L} V_t^\varphi\, \pi(\mathsf{d}\varphi) = \beta \int_{\Phi_L} \int_{(-\infty,t]} \mathsf{e}^{-X_t^\varphi - X_s^\varphi}\, \mathsf{d}s\, \pi(\mathsf{d}\varphi)$$

where π is a probability measure on Φ_L. All COGARCH processes V^φ are driven by the same Lévy process L.

In order to ensure that $\bar{V}_t^{(2)}$ is finite, we assume $\int_{\Phi_L} V_0^\varphi\, \pi(\mathsf{d}\varphi) < \infty$.

If $\pi = \sum_{i=1}^\infty p_i \delta_{\varphi_i}$, we obviously have $\bar{V}^{(2)} = \sum_{i=1}^\infty p_i V^{\varphi_i}$ with dependent summands.

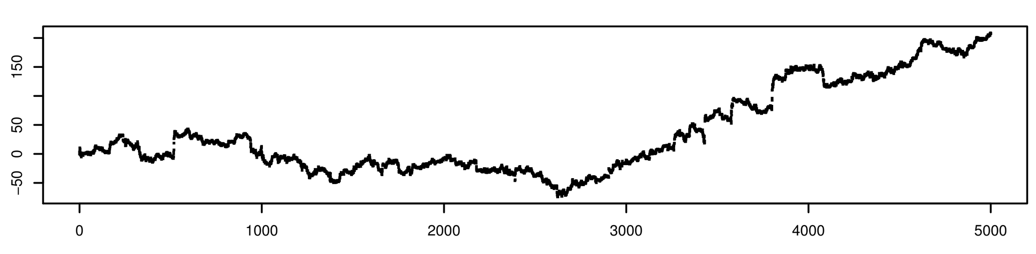
Main features:

- The jumps of the supCOGARCH 2 are given by $\Delta\bar{V}_t^{(2)} = \int_{\Phi_L} \varphi V_{t-}^\varphi\, \pi(\mathsf{d}\varphi) \Delta S_t, \quad t \geq 0.$
- The process $(\bar{V}_t^{(2)})_{t\in\mathbb{R}}$ is strictly stationary.
- The stationary distribution is uniquely determined by the law of $\int_{\Phi_L} V_\infty^\varphi\, \pi(\mathsf{d}\varphi) = \beta \int_{\Phi_L} \int_{\mathbb{R}_+} \mathsf{e}^{-X_s^\varphi}\, \mathsf{d}s\, \pi(\mathsf{d}\varphi)$.
- First and second moment of the stationary process are given by
$\mathbb{E}[\bar{V}_t^{(2)}] = \int_{\Phi_L} \mathbb{E}[V_0^\varphi]\, \pi(\mathsf{d}\varphi) = \int_{\Phi_L} \frac{\beta}{\eta - \varphi\mathbb{E}[S_1]}\, \pi(\mathsf{d}\varphi) \leq \infty$, and
$\mathsf{Cov}\,[\bar{V}_t^{(2)}, \bar{V}_{t+h}^{(2)}]$
$= \iint_{\Phi_L^2} \mathsf{e}^{h\Psi(1,\tilde{\varphi})} \mathsf{Cov}\,[V_0^\varphi, V_0^{\tilde{\varphi}}]\, \pi(\mathsf{d}\varphi)\, \pi(\mathsf{d}\tilde{\varphi}) \leq \infty$ where
$\mathsf{Cov}\,[V_0^\varphi, V_0^{\tilde{\varphi}}] = \frac{\beta^2 \varphi\tilde{\varphi} \mathsf{Var}\,[S_1]}{(\varphi\mathbb{E}[S_1]-\eta)(\tilde{\varphi}\mathbb{E}[S_1]-\eta)(2\eta-(\varphi+\tilde{\varphi})\mathbb{E}[S_1]-\varphi\tilde{\varphi}\mathsf{Var}\,[S_1])}$.
- Typically the stationary distribution has Pareto-like tails.
- $\bar{V}^{(2)}$ is no Markov process with respect to its natural filtration.

The **integrated supGOGARCH 2 process** is naturally defined as

$$G_t^{(2)} := \int_{(0,t]} \sqrt{\bar{V}_{s-}^{(2)}}\, \mathsf{d}L_s, \quad t \geq 0.$$

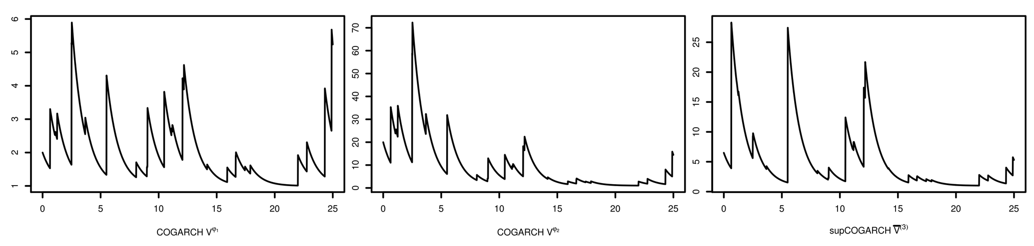
- $G^{(2)}$ jumps exactly at the times when $\bar{V}^{(2)}$ jumps.
- $(G_t^{(2)})_{t\in\mathbb{R}}$ has stationary increments if $(\bar{V}_t^{(2)})_{t\in\mathbb{R}}$ is strictly stationary.
- The integrated supCOGARCH 2 process has the same second-order structure as the integrated supCOGARCH 1 and, hence, as the integrated COGARCH.



Sample path of an integrated supCOGARCH 2 process.

The supCOGARCH 3

Lévy field based approach



Sample paths of two dependent COGARCH processes and the resulting supCOGARCH 3 process.

Let Λ^L be a Lévy basis on $\mathbb{R} \times \Phi_L$ such that $L_t := \Lambda^L((0,t] \times \Phi_L)$, $t \geq 0$ and $L_t := -\Lambda^L((-t,0] \times \Phi_L)$, $t < 0$ exists for every $t \in \mathbb{R}$. Set $\Lambda^S := [\Lambda^L, \Lambda^L]^{\mathsf{d}}$ and define $S_t := \Lambda^S((0,t] \times \Phi_L)$, $t \geq 0$, $S_t := -\Lambda^S((-t,0] \times \Phi_L)$, $t < 0$. Assume that the characteristics of Λ^S are given by $(0, 0, \mathsf{d}t\, \nu_S(\mathsf{d}y)\, \pi(\mathsf{d}\varphi))$ for some probability measure π on Φ_L.

The **supCOGARCH 3 volatility process** $\bar{V}^{(3)}$ is defined for $t \geq 0$ by
$\bar{V}_t^{(3)} = \bar{V}_0^{(3)} + \beta t - \eta \int_{(0,t]} \bar{V}_s^{(3)}\, \mathsf{d}s + \int_{(0,t]} \int_{\Phi_L} \varphi V_{s-}^\varphi\, \Lambda^S(\mathsf{d}s, \mathsf{d}\varphi)$,
where $\bar{V}_0^{(3)}$ is some starting random variable independent of the restriction of Λ^L to $\mathbb{R}_+ \times \Phi_L$. For every $\varphi \in \Phi_L$, V^φ denotes the two-sided COGARCH process driven by S.

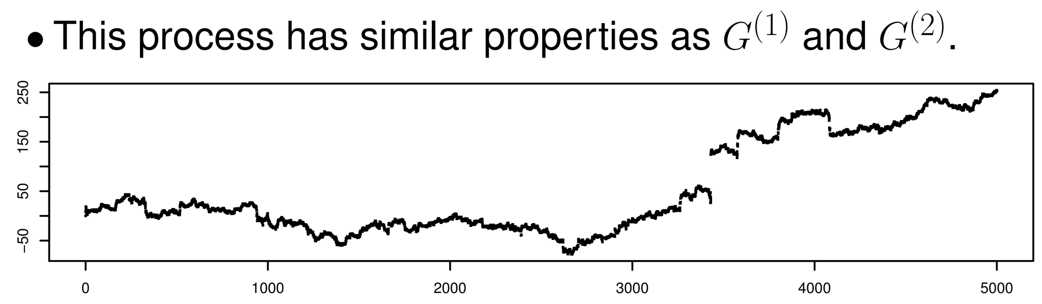
Main features:

- The jumps of the supCOGARCH 3 are given by $\Delta\bar{V}_t^{(3)} = \int_{\mathbb{R}_+\times\Phi_L} \varphi V_{t-}^\varphi y\, \mu^{\Lambda^S}(\{t\}, \mathsf{d}\varphi, \mathsf{d}y), \quad t \geq 0,$ where μ^{Λ^S} is the jump measure induced by Λ^S.
- $\bar{V}_t^{(3)} = \mathsf{e}^{-\eta t}\left(\bar{V}_0^{(3)} + \frac{\beta}{\eta}(\mathsf{e}^{\eta t} - 1) + \int_{(0,t]} \mathsf{e}^{\eta s}\, \mathsf{d}A_s\right), \quad t \geq 0,$ where $A_t := \int_{(0,t]} \int_{\Phi_L} \varphi V_{s-}^\varphi\, \Lambda^S(\mathsf{d}s, \mathsf{d}\varphi)$, $t \geq 0$, is a semi-martingale with increasing sample paths.
- $\bar{V}^{(3)}$ has a strictly stationary distribution if and only if $\int_{\mathbb{R}_+} \int_{\Phi_L} \int_{\mathbb{R}_+} 1 \wedge (y\varphi V_s^\varphi \mathsf{e}^{-\eta s})\, \mathsf{d}s\, \pi(\mathsf{d}\varphi)\, \nu_S(\mathsf{d}y) < \infty$ a.s.
- The stationary distribution is uniquely determined by the law of $\frac{\beta}{\eta} + \int_{\mathbb{R}_+} \mathsf{e}^{-\eta s}\, \mathsf{d}A_s$.
- First and second moment of the stationary process are given by
$\mathbb{E}[\bar{V}_t^{(3)}] = \int_{\Phi_L} \mathbb{E}[V_0^\varphi]\, \pi(\mathsf{d}\varphi) = \int_{\Phi_L} \frac{\beta}{\eta - \mathbb{E}[S_1]\varphi}\, \pi(\mathsf{d}\varphi) \leq \infty$ and
$\mathsf{Cov}\,[\bar{V}_t^{(3)}, \bar{V}_{t+h}^{(3)}] = \iint_{\Phi_L^2} \Big(\mathsf{e}^{h\Psi(1,\varphi)} \mathsf{Cov}\,[V_0^\varphi, V_0^{\tilde{\varphi}}]$
$+ \mathsf{e}^{-\eta h} \frac{\beta}{\eta} \frac{\mathsf{Var}\,[V_0^\varphi] - \mathsf{Cov}\,[V_0^\varphi, V_0^{\tilde{\varphi}}]}{\mathbb{E}[V_0^\varphi]}\Big)\, \pi(\mathsf{d}\tilde{\varphi})\, \pi(\mathsf{d}\varphi) \leq \infty.$
- Typically the stationary distribution has Pareto-like tails.
- $\bar{V}^{(3)}$ is no Markov process with respect to its natural filtration.

The **integrated supGOGARCH 3 process** is naturally defined as

$$G_t^{(3)} := \int_{(0,t]} \sqrt{\bar{V}_{s-}^{(3)}}\, \mathsf{d}L_s, \quad t \geq 0.$$

- This process has similar properties as $G^{(1)}$ and $G^{(2)}$.



Sample path of an integrated supCOGARCH 3 process.

Main References

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