On infinitely divisible semimartingales

Jan Rosiński, University of Tennessee

Joint work with Andreas Basse-O'Connor

7th International Conference on Lévy Processes: Theory and Applications
July 15 - 19, 2013, Wrocław

Outline

$\mathbb{F} = \{\mathcal{F}_t\}_{t \geq 0}$ a filtration of $(\Omega, \mathcal{F}, \mathbb{P})$.
Recall that a process $X = (X_t)_{t \geq 0}$ is called a semimartingale with respect to $\mathbb{F}$ if it admits a decomposition

$$X_t = X_0 + M_t + A_t \tag{1}$$

where $(M_t)_{t \geq 0}$ is a local martingale with respect to $\mathbb{F}$ and $(A_t)_{t \geq 0}$ is an $\mathbb{F}$-adapted process with càdlàg paths of finite variation and such that $A_0 = M_0 = 0$.

X is called a special semimartingale if, in addition, A in (1) can be chosen predictable; in this case the decomposition (1) is unique and it is called the canonical decomposition of X.

Every Lévy process X is a semimartingale; it is special $\iff$ $\mathbb{E}|X_1| < \infty$.

Theorem (Stricker (1983))

Let $X = (X_t)_{t \geq 0}$ be a càdlàg Gaussian process. Then X is a semimartingale if and only

$$X_t = X_0 + M_t + A_t$$

where M is a Gaussian process with independent increments and A is a Gaussian process of finite variation.

WE CONSIDER GENERALIZATIONS OF STRICKER'S THEOREM TO INFINITELY DIVISIBLE (ID) PROCESSES.

Example (ID martingales)

Let $X_t = \sum_{k=1}^{N} \xi_k(t)$, where $(\xi_k)_{k\geq 0}$ are iid copies of a process $(\xi(t))_{t\geq 0}$. Then $(X_t)_{t\geq 0}$ is a martingale if and only if $(\xi_t)_{t\geq 0}$ is.

Proof: Let $0 \leq s_1 < \cdots < s_n = s < t$ and $u_1, \ldots, u_n \in \mathbb{R}$.

$$\mathbb{E}\Big[(X_t - X_s)e^{i\sum_{j=1}^{n} u_j X_{s_j}}\Big] = \frac{1}{i}\frac{\partial}{\partial\theta}\mathbb{E}\Big[e^{i\theta(X_t - X_s) + i\sum_{j=1}^{n} u_j X_{s_j}}\Big]_{\theta=0}$$

$$= \frac{1}{i}\frac{\partial}{\partial\theta}\exp\left[\mathbb{E}\Big(e^{i\theta(\xi_t - \xi_s) + i\sum_{j=1}^{n} u_j \xi_{s_j}} - 1\Big)\right]_{\theta=0}$$

$$= \exp\left[\mathbb{E}\Big(e^{i\sum_{j=1}^{n} u_j \xi_{s_j}} - 1\Big)\right]\mathbb{E}\Big[(\xi_t - \xi_s)e^{i\sum_{j=1}^{n} u_j \xi_{s_j}}\Big].$$

Hence $\mathbb{E}[X_t - X_s | \mathcal{F}_s^X] = 0 \iff \mathbb{E}[\xi_t - \xi_s | \mathcal{F}_s^\xi] = 0$. $\square$

Corollary:

We cannot expect a full extension of Stricker's thm to the ID case.

Example (ID property of the decomposition)

Let Y and Z be independent random variables, where Y is double exponential and Z standard normal. Put

$$X_t = \begin{cases} Y + Z & 0 \le t < 1 \\ Y & t \ge 1. \end{cases}$$

Then (X_t) is a special semimartingale relative to the natural filtration with the canonical decomposition

$$M_t = \begin{cases} 0 \\ \mathbb{E}[Z|Y+Z] - Z \end{cases} \quad \text{and } A_t = \begin{cases} 0 & , t < 1 \\ -\mathbb{E}[Z|Y+Z] & , t \ge 1 \end{cases}$$

such that M and A are not ID.

Example (Continue)

Indded, one can show that $g(Y+Z) = \mathbb{E}[Z|Y+Z]$ is bounded, so it cannot be ID.

Then $Z - g(Y+Z)$ has Gaussian integrability but it is not Gaussian. Thus it cannot be ID.

Example (Embedding X into the ID noise)

Let $U^1 = (U^1_t)_{t\in\mathbb{R}}$ be a symmetrized Gamma Lévy process with scale and shape parameters 1 and $U^2 = (U^2_t)_{t\in\mathbb{R}}$ be a standard Brownian motion such that U^1 and U^2 are independent. Let $V = \{1,2\}$ and for all $a < b$ and $B \subseteq \{1,2\}$ set

$$\Lambda((a,b] \times B) = \sum_{i\in B} (U^i_b - U^i_a).$$

Then Λ is an independently scattered random measure. Let

$$\phi(t,s,v) = \begin{cases} \mathbf{1}_{[-1,0]\times\{1,2\}}(s,v) & t < 1 \\ \mathbf{1}_{[-1,0]\times\{1\}}(s,v) & t \geq 1. \end{cases}$$

Then $X = (X_t)_{t\geq 0}$ of the previous example is equal to

$$X_t = \int_{(-\infty,t]\times V} \phi(t,s,v)\,\Lambda(ds,dv).$$

Example (continue)

Process X is a semimartingale with respect to the filtration $\mathcal{F}^\Lambda$ the canonical decomposition $X_t = X_0 + A_t + M_t$ of X is given by

$$M_t \equiv 0, \qquad A_t = \begin{cases} 0 & t < 1 \\ U_0^2 - U_{-1}^2 & t \geq 1, \end{cases}$$

In this setting decomposition is into ID processes.

Hida's multiplicity representation of Gaussian processes - simplified version:

Proposition

Let $X = (X_t)_{t\in\mathbb{R}}$ be a centered Gaussian process which is right-continuous in probability. Then there is an independently scattered Gaussian random measure Λ on $\mathbb{R}_+ \times V$, where $V \subset \mathbb{Z}_+$, and a family of Λ-integrable functions $\phi(t,\cdot,\cdot) : \mathbb{R} \times V \to \mathbb{R}$ with $\phi(t,s,\cdot) = 0$ whenever $s > t$ such that

$$X_t = \int_{[0,t]\times V} \phi(t,s,v)\,\Lambda(ds,dv)$$

and $\mathcal{F}_t^X = \mathcal{F}_t^\Lambda$ for every $t \geq 0$.

$(V, \mathcal{V})$ is a measurable space.

$\tilde{V} = \mathbb{R} \times V$ and $\tilde{\mathcal{V}} = \mathcal{B}(\mathbb{R}) \otimes \mathcal{V}$.

Given a (feasible) sequence $\{V_n\} \subset \mathcal{V}$, $V_n \uparrow V$, consider

$$\mathcal{S} = \{A \in \tilde{\mathcal{V}} : A \subset [-n, n] \times V_n \text{ for some } n \geq 1\}.$$

Let Λ a mean-zero independently scattered random measure on $\mathcal{S}$ such that for every $A \in \mathcal{S}$, $\Lambda(A)$ has an infinitely divisible distribution determined by

$$\log \mathbb{E} e^{i\theta\Lambda(A)} = \int_A \Big[i\theta b(u) - \frac{1}{2}\theta^2\sigma^2(u) + \int_{\mathbb{R}} \left(e^{i\theta x} - 1 - i\theta\tau(x)\right) \rho_u(dx) \Big] \kappa(du), \tag{2}$$

where $u = (s, v) \in \mathbb{R} \times V$, $b\colon \mathbb{R} \times V \to \mathbb{R}$ is a measurable function, κ is a σ-finite measure on $\mathbb{R} \times V$, $\{\rho_u\}_{u \in \mathbb{R} \times V}$ is a measurable family of Lévy measures on $\mathbb{R}$, and τ is a (fixed) continuous truncation function.

Assume

$$\kappa(\{t\} \times V) = 0 \quad \text{for every } t \in \mathbb{R}. \tag{3}$$

We consider ID processes $\mathbf{X} = (X_t)_{t \geq 0}$ of the form

$$X_t = \int_{(-\infty,t]\times V} \phi(t,s,v)\,\Lambda(ds,dv),$$

$\phi : \mathbb{R}^2 \times V \mapsto \mathbb{R}$ is such that $\phi(t,\cdot,\cdot)$ is Λ-integrable for $t \geq 0$, and $\phi(\cdot,s,v)$ is càdlàg for each (s,v) with $\phi(t,s,v) = 0$ if $t < s$.

Λ is a random measure specified in (2)-(3).

$$\mathcal{F}_t^{\Lambda} = \sigma\{A \in \mathcal{S} : A \subset (-\infty,t] \times V\}, \quad t \geq 0.$$

EXAMPLES:

Fractional processes:

$$X_t = \int_{\mathbb{R}} \left[(t-s)_+^{\alpha} - (-s)_+^{\alpha} \right] dZ_s$$

where $(Z_t)_{t\in\mathbb{R}}$ is a two-sided Lévy process.
If Z is a Brownian motion, then $\alpha = H - 1/2$, where H is the Hurst index. $(X_t)_{t\geq 0}$ is never a semimartingale unless $H = 1/2$ in which case X is a Brownian motion.

Is X a semimartingale for some Lévy process Z?

Moving average:

$$X_t = \int_{-\infty}^{t} f(t-s)\, dZ_s$$

where f is càdlàg. Example, Ornstein-Uhlenbeck process, $f(x) = e^{-\lambda x}$.

Mixed moving averages. For example, supOU process:

$$X_t = \int_{(-\infty,t]\times\mathbb{R}_+} e^{-\lambda(t-s)}\, W(ds\, d\lambda)$$

SIMA processes:

$$X_t = \int_{\mathbb{R}} \left[f(t-s) - f_0(-s) \right] dZ_s$$

where f is càdlàg with $f(x) = f_0(x) = 0$ for $x < 0$.

SIMMA processes:

$$X_t = \int_{\mathbb{R}\times V} \left[f(t-s, v) - f_0(-s, v) \right] W(ds\, dv)$$

When such processes are semimartingales?
If they are semimartingales, then what characterize their martingale and finite variation parts.

Theorem

Let $\mathbf{X} = (X_t)_{t \geq 0}$ *be a process of the form*

$$X_t = \int_{(-\infty,t]\times V} \phi(t,s,v)\,\Lambda(ds,dv),$$

as above. Then $\mathbf{X}$ *is a semimartingale with respect to the filtration* $\mathbb{F}^{\Lambda} = (\mathcal{F}_t^{\Lambda})_{t\geq 0}$ *if and only if*

$$X_t = X_0 + M_t + A_t, \quad t \geq 0, \tag{4}$$

where $\mathbf{M} = (M_t)_{t\geq 0}$ *is a continuous in probability semimartingale with independent increments given by*

$$M_t = \int_{(0,t]\times V} \phi(s,s,v)\,\Lambda(ds,dv), \quad t \geq 0 \tag{5}$$

Theorem (Continue)

(i.e., the integral exists), and $\mathbf{A} = (A_t)_{t\geq 0}$ *is a predictable càdlàg process of finite variation given by*

$$A_t = \int_{(-\infty,t]\times V} [\phi(t,s,v) - \phi(s_+,s,v)]\,\Lambda(ds,dv). \qquad (6)$$

Decomposition (4) *is unique in the following sense: If* $\mathbf{X} = X_0 + \mathbf{M}' + \mathbf{A}'$, *where* $\mathbf{M}'$ *is a continuous in probability semimartingale with independent increments and* $\mathbf{A}'$ *is a predictable càdlàg process of finite variation, then* $\mathbf{M}' = \mathbf{M} + g$ *and* $\mathbf{A}' = \mathbf{A} - g$ *for some continuous deterministic function* g *of finite variation, with* $\mathbf{M}$ *and* $\mathbf{A}$ *given by* (5) *and* (6).

Theorem (Continue)

If $\mathbf{X}$ *is a semimartingale, then it is a special semimartingale if and only if* $\mathbb{E}|M_t| < \infty$ *for all* $t > 0$*; and in this case* $(M_t - \mathbb{E}M_t)_{t\geq 0}$ *is a martingale and*

$$X_t = X_0 + (M_t - \mathbb{E}M_t) + (A_t + \mathbb{E}M_t), \quad t \geq 0$$

is the canonical decomposition of $\mathbf{X}$*.*

For simplicity assume that Λ is a symmetric random measure without Gaussian component, i.e. $\sigma^2 \equiv 0$ and

$$\eta_y(x,\infty) = \eta_y(-\infty,-x) \quad x > 0$$

in which case (2) becomes

$$\mathbb{E}e^{iu\Lambda(A)} = \exp\Big[-2\int_A\int_0^\infty (1-\cos ux)\,\eta_y(dx)\,\kappa(dy)\Big].$$

Choose a probability measure $\tilde{\kappa}$ on $\tilde{V}$ and let

$$h(y) = \frac{1}{2}\frac{d\tilde{\kappa}}{d\kappa}(y).$$

By JR (2001) there exist there independent sequences $\{\Gamma_i\}$, $\{\epsilon_i\}$, $\{T_i\}$ on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with such that

- Γ_i are the partial sums of iid standard exponential random variables,
- ϵ_i are symmetric Bernoulli random variables,
- $T_i = (T_i^1, T_i^2)$ are iid random variables in $\tilde{V} = \mathbb{R} \times V$ with the common distribution $\tilde{\kappa}$,

such that

$$\Lambda(A) = \sum_{i=1}^{\infty} R_i \mathbf{1}_A(T_i) \quad a.s.,\ A \in \mathcal{S},$$

where

$$R_i = \epsilon_i R(\Gamma_i h(T_i), T_i) \qquad \text{(dependent symmetric sequence)}$$

with

$$R(u, y) = \inf\{x > 0 : \eta_y(x, \infty) \le u\}, \quad u > 0.$$

Theorem

[1] *Let* $\mathbf{X} = (X_t)_{t\in\mathbb{R}}$ *be a càdlàg process given by*

$$X_t = \int_{\mathbb{R}\times V} \phi(t,s,v)\,\Lambda(ds,dv),$$

where $\phi : \mathbb{R}^2 \times V \mapsto \mathbb{R}$ *is such that* $\phi(t,\cdot,\cdot)$ *is* Λ*-integrable for each* t*, and* $\phi(\cdot,s,v)$ *is càdlàg for each* (s,v)*. Then*

$$\sum_{i=1}^{\infty} R_i \phi(t, T_i) = X_t \quad a.s. \tag{7}$$

where the series converges uniformly in t *on compacts. Consequently, the processes in* (7) *are indistinguishable as are the processes*

$$\sum_{i=1}^{\infty} R_i \Delta\phi(t, T_i) = \Delta X_t \,. \tag{8}$$

[1] **Basse-O'Connor & JR: AoP (2013), to appear.**

Let X be as in the Theorem. Since the distribution of T_j^1 is continuous by assumption (3),

$$\Delta\phi(T_j^1, T_i) = 0 \quad a.s. \; i \neq j.$$

Thus from (8),

$$\Delta X_{T_j^1} = R_j \Delta\phi(T_j^1, T_j) = R_j \phi(T_j^1, T_j)$$

The last equality holds because $\phi(t, s, v) = 0$ if $t < s$.

Assuming that X is a semimartingale we get

$$\begin{aligned}\infty > \sum_{s\le t}(\Delta X_s)^2 &\ge \sum_{j=1}^{\infty}(\Delta X_{T_j^1})^2 \mathbf{1}_{(0,t]}(T_j^1) \\ &= \sum_{j=1}^{\infty} R_j^2 \phi(T_j^1, T_j)^2 \mathbf{1}_{(0,t]}(T_j^1)\,.\end{aligned}$$

By the symmetry of $\{R_j\}$, M_t defined as

$$M_t := \sum_{j=1}^{\infty} R_j \phi(T_j^1, T_j) \mathbf{1}_{(0,t]}(T_j^1) = \int_{(0,t]\times V} \phi(s,s,v)\,\Lambda(ds,dv)$$

exists, it is a symmetric process with independent increments, and can also be written as above stochastic integral.

Since M is a semimartingale, it is special if and only if $J_t = \sup_{s \leq t} |\Delta M_s|$ is locally integrable.
We have

$$J_t = \sup_{s \leq t} |\Delta M_s| = \sup_j |R_j \phi(T_j^1, T_j) \mathbf{1}_{(0,t]}(T_j^1)|$$
$$\leq \sup_{s \leq t} |\Delta X_s|$$

where the last term is locally integrable when X is special. This M is special as well.

Thus M is a special semimartingale, and since is an independent increment process, it must have finite expectation, necessarily zero.

To show that the process

$$A_t = X_t - M_t$$

is predictable with finite variation, it is enough to prove the uniqueness of M as the martingale part in the decomposition (1) of X.

This is possible by a close examination of the form of jumps of local martingales with respect to $\mathbb{F}^{\Lambda}$.

Let $B \in \mathcal{V}_0$, a countable ring generating $\mathcal{V}$.

$$Z_t^B := \Lambda((0,t] \times B) \quad t \geq 0$$

is a process with independent increments and stochastically continuous because (3). We have

$$\mathcal{F}_t^{\Lambda} = \mathcal{F}_0^{\Lambda} \vee \sigma\Big((Z_t^B)_{t\geq 0} : B \in \mathcal{V}_0\Big).$$

Proposition

Any local martingale $(N_t)_{t\geq 0}$ with respect to $\mathcal{F}_t^{\Lambda}$ is purely discontinuous and

$$\{\Delta N_t : t \geq 0\} \subset \{\Delta Z_t^B : t \geq 0,\ B \in \mathcal{V}_0\} \quad a.s.$$

Moreover, jumps of Z_t^B can be represented by a sequence of totally inaccessible stopping times.

On the other hand, applying (8) for $\phi(t,s,v) = \mathbf{1}_{(0,t]}(s)\mathbf{1}_B(v)$ we get

$$\Delta Z_t^B = \sum_{i=1}^{\infty} R_i \mathbf{1}_{\{t\}}(T_i^1)\mathbf{1}_B(T_i^2) \quad a.s.$$

Corollary

There exists a sequence of totally inaccessible stopping times $\{\tau_k\}$ *such that*

$$\{\tau_k : k \in \mathbb{N}\} = \{\Delta Z_t^B : t \geq 0,\ B \in \mathcal{V}_0\} \subset \{T_i^1 : i \in \mathbb{N}\}.$$

Consequently, any local martingale $(N_t)_{t\geq 0}$ *with respect to* $\mathcal{F}_t^\Lambda$ *is purely discontinuous and*

$$\{\Delta N_t : t \geq 0\} \subset \{\tau_k : k \in \mathbb{N}\} \subset \{T_i^1 : i \in \mathbb{N}\} \quad a.s.$$

Uniqueness of M:

Let N be the local martingale and F a predictable process of finite variation in the decomposition

$$X_t = X_0 + N_t + F_t$$

We have

$$\begin{aligned}
\Delta N_{\tau_k} &= \Delta X_{\tau_k} - \Delta F_{\tau_k} = \Delta X_{\tau_k} \text{ (F is predictable and τ_k totally inaccessible)} \\
&= \Delta X_{T_j^1} \text{ ($\tau_k = T_j^1$ for some random j)} \\
&= \Delta M_{T_j^1} \text{ (from the construction of M)} \\
&= \Delta M_{\tau_k}.
\end{aligned}$$

Since both M and N are purely discontinuous local martingales with equal jumps,

$$M = N.$$

□

Using A. Basse-O'Connor & J.R. SPA 123, (2013)

Consider a process $\mathbf{X} = (X_t)_{t \in \mathbb{R}}$ of the form

$$X_t = \int_{\mathbb{R}\times V} \left[f(t-s,v) - f_0(-s,v) \right] W(ds,dv), \qquad t \in \mathbb{R}, \quad (9)$$

where W is an independently scattered random measure defined on $\mathcal{S}$ such that for every $A \in \mathcal{S}$, $W(A)$ is infinitely divisible with the characteristic function

$$\mathbb{E}e^{iuW(A)} = \exp\Big[\int_A \Big(iub(v) - \frac{1}{2}u^2\sigma^2(v) + \int_{\mathbb{R}} (e^{iux} - 1 - iu\tau(x)\,\rho_v(dx) \Big)\, ds\, m(dv) \Big].$$

Here $\{\rho_v\}_{v \in V}$ is a measurable parametrization of Lévy measures on $\mathbb{R}$, b and σ^2 are two measurable functions from V into $\mathbb{R}$ with $\sigma^2 \geq 0$, and τ is a continuous truncation function.

The process $\mathbf{X}$ is infinitely divisible, has stationary increments and is continuous in probability.

The functions f and f_0 are measurable and deterministic.

If V is a one-point space (or simply, there is no v-component in (9)) and $f_0 = 0$, then (9) defines a moving average (a mixed moving average for a general V). If V is a one-point space and $f_0(x) = f(x) = x_+^{\alpha}$ for some $\alpha \in \mathbb{R}$, then $\mathbf{X}$ is a fractional Lévy process.

We consider a SIMMA process $\mathbf{X} = (X_t)_{t\geq 0}$ given by (9), where for each $v \in V$, $f(\cdot, v)$ and $f_0(\cdot, v)$ are càdlàg with $f(s, v) = f_0(s, v) = 0$ for all $s < 0$. Let $\mathbb{F}^W = \{\mathcal{F}_t^W\}_{t\geq 0}$ be the natural filtration generated by W,

$$\mathcal{F}_t^W = \sigma\Big(W(A) : A \in \mathcal{S},\, A \subseteq (-\infty, t] \times V\Big).$$

Theorem (Sufficiency)

Suppose that for m-a.e. $v \in V$, $f(\cdot, v)$ is absolutely continuous on $[0, \infty)$ with derivative $\dot{f}(s, v) = \frac{\partial}{\partial s} f(s, v)$ satisfying

$$\int_V \int_0^\infty \left(|\dot{f}(s,v)|^2 \sigma^2(v)\right) ds\, m(dv) < \infty \tag{10}$$

$$\int_V \int_0^\infty \int_{\mathbb{R}} \left(|x\dot{f}(s,v)| \wedge |x\dot{f}(s,v)|^2\right) \rho_v(dx)\, ds\, m(dv) < \infty. \tag{11}$$

Then X is a semimartingale with respect to $\mathbb{F}^W$.

Theorem (Necessity)

Suppose that X *is a semimartingale with respect to* $\mathbb{F}$ *and*

$$m\Big(v \in V : \int_{-1}^{1} |x|\, \rho_v(dx) < \infty,\, \sigma^2(v) = 0\Big) = 0.$$

Then for m*-a.e.* v*,* $f(\cdot, v)$ *is absolutely continuous on* $[0, \infty)$ *with derivative* $\dot{f}(\cdot, v)$ *satisfies* (10) *and*

$$\int_0^\infty \int_{\mathbb{R}} \Big(\frac{|x\dot{f}(s,v)| \wedge |x\dot{f}(s,v)|^2}{x^2 \vee 1}\Big)\, \rho_v(dx)\, ds < \infty.$$

Furthermore, if there exist two measurable functions $u_0 \colon V \to [0, \infty)$ *and* $K_0 \colon V \to (0, \infty)$ *such that for* m*-a.e.* v*,*

$$u \int_{|x|>u} |x|\, \rho_v(dx) \le K_0(v) \int_{|x|\le u} x^2\, \rho_v(dx), \qquad \text{for all } u > u_0(v),$$

Theorem (Necessity; continue)

then for m-a.e. v,

$$\int_0^\infty \int_{\mathbb{R}} (|x\dot{f}(s,v)|^2 \wedge |x\dot{f}(s,v)|)\, \rho_v(dx)\, ds < \infty,$$

and if $u_0 \equiv 0$ *and* $\mathrm{esssup}_V K_0 < \infty$, *then* (10)–(11) *is satisfied.*

Remark

The assumption with $K_0 > 0$ always holds when W is a random measure with finite variance. Actually, we do not know an example when this assumption fails.

Corollary (Fractional Lévy processes)

Let $\gamma > 0$, $x_+ := \max\{x, 0\}$ *for* $x \in \mathbb{R}$, $\mathbf{Z}$ *be a Lévy process as above, and* $\mathbf{X}$ *be a fractional Lévy process defined by*

$$X_t = \int_{-\infty}^{t} \{(t-s)_+^{\gamma} - (-s)_+^{\gamma}\} \, dZ_s$$

where the stochastic integrals exist. Then $\mathbf{X}$ *is a semimartingale with respect to* $\mathbb{F}^Z$ *if and only if* $\sigma^2 = 0$, $\gamma \in (0, \frac{1}{2})$ *and*

$$\int_{\mathbb{R}} |x|^{\frac{1}{1-\gamma}} \rho(dx) < \infty.$$

Corollary

Suppose that $\mathbf{Z} = (Z_t)_{t\in\mathbb{R}}$ *is a two-sided Lévy process as above, with paths of infinite variation on compact intervals. Let* $\mathbf{X} = (X_t)_{t\geq 0}$ *be a process of the form*

$$X_t = \int_{-\infty}^{t} \{f(t-s) - f_0(-s)\}\, dZ_s.$$

Suppose that the random variable Z_1 *is either square-integrable or has a regularly varying distribution at* ∞ *of index* $\beta \in [-2, -1)$*. Then* $\mathbf{X}$ *is a semimartingale with respect to* $\mathbb{F}^Z$ *if and only if* f *is absolutely continuous on* $[0, \infty)$ *with a derivative* $\dot{f}$ *satisfying*

$$\int_0^\infty |\dot{f}(t)|^2\, dt < \infty \qquad \text{if } \sigma^2 > 0,$$

$$\int_0^\infty \int_{\mathbb{R}} \left(|x\dot{f}(t)| \wedge |x\dot{f}(t)|^2\right) \rho(dx)\, dt < \infty.$$

Example (Tempered stable)

Suppose that $\mathbf{Z}$ is a symmetric tempered stable Lévy process with indexs $\alpha \in [1,2)$ and $\lambda > 0$, i.e., $\rho(dx) = c|x|^{-\alpha-1}e^{-\lambda|x|}\,dx$ where $c > 0$, and $\sigma^2 = b = 0$. Then $\mathbf{X}$ is a semimartingale with respect to $\mathbb{F}^Z$ if and only if f is absolutely continuous on $[0,\infty)$ with a derivative $\dot{f}$ satisfying

$$\int_0^\infty \left(|\dot{f}(t)|^\alpha \wedge |\dot{f}(t)|^2\right) ds < \infty.$$

Thank you!