Anticipating Girsanov identities and the laws of stopping sets

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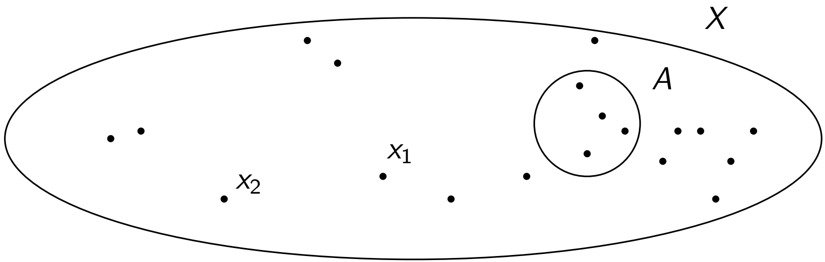
7th International Conference on Lévy Processes

15-19 July 2013, Wrocław

Poisson (1781-1840) random measures



(X, σ) a measure space, $\Omega^X = \left\{ \omega = \sum_{k \geq 1} \varepsilon_{x_k} \right\}$ the space of configurations on X,

$$\mathbb{P}_\sigma(\omega \in \Omega^X \ : \ \omega(A) = n) = e^{-\sigma(A)} \frac{(\sigma(A))^n}{n!}, \qquad n \in \mathbb{N},$$

where $\omega(A) = \#\{k \ : \ x_k \in A\}$.

Poisson space

$$\Omega^X = \{\omega = (x_i)_{i=1,\dots,N} \subset X,\ x_i \neq x_j\ \forall i \neq j,\ N \in \mathbb{N} \cup \{\infty\}\}$$

is endowed with the Poisson probability measure π_σ on X with intensity $\sigma(dx)$ on X, such that for all compact disjoint subsets $A_1, \dots, A_n$ of X, $n \geq 1$, the mapping

$$\omega \mapsto (\omega(A_1), \dots, \omega(A_n))$$

is a vector of independent Poisson distributed random variables on $\mathbb{N}$ with respective intensities $\sigma(A_1), \dots, \sigma(A_n)$.

Each element ω of Ω^X is identified to

$$\omega = \sum_{x \in \omega} \epsilon_{x_i},$$

where ϵ_x denotes the Dirac measure at $x \in X$, and we have

$$\int_X f(x)\omega(dx) = \sum_{x \in \omega} f(x).$$

Multiple stochastic integrals

Given $f_n \in L^2(X^n)$ a symmetric function, let

$$I_n(f_n) = \int_{X^n} f_n(x_1, \ldots, x_n) \mathbf{1}_{\{x_i \neq x_i \ : \ i \neq j\}} (\omega(dx_1) - \sigma(dx_1)) \cdots (\omega(dx_n) - \sigma(dx_n)),$$

with the Itô isometry

$$E[I_n(f_n) I_m(g_m)] = n! \mathbf{1}_{\{n=m\}} \langle f_n, g_m \rangle_{L^2(X^n, \sigma^{\otimes n})}.$$

The exponential vector

$$\xi(f) := 1 + \sum_{n=1}^{\infty} \frac{1}{n!} I_n(f^{\otimes n})$$

satisfies

$$\xi(f) = \exp\left(- \int_X f(x) \sigma(dx) \right) \prod_{x \in \omega} (1 + f(x)).$$

Difference operator

The finite difference operator

$$D_x F(\omega) = F(\omega \cup \{x\}) - F(\omega)$$

satisfies

$$D_x \xi(f) = (1 + f(x))\xi(f) - \xi(f) = f(x)\xi(f)$$

and

$$D_x I_n(f_n) := n I_{n-1}(f_n(*, x)), \qquad x \in X. \tag{1}$$

Given F written as

$$F = E[F] + \sum_{k=1}^{\infty} I_k(g_k)$$

we have

$$D_{x_1} \cdots D_{x_n} F = \sum_{k=n}^{\infty} k! I_{k-n}(g_k(*, x_1, \ldots, x_n)),$$

hence

$$g_n(x_1, \ldots, x_n) = \frac{1}{n!} E[D_{x_1} \cdots D_{x_n} F].$$

$\mathcal{S}$-transform

Given F written as

$$F = E[F] + \sum_{k=1}^{\infty} I_k(g_k),$$

we have

$$\begin{aligned} E\left[I_n(f^{\otimes n})F\right] &= n!\langle f^{\otimes n}, g_n\rangle \\ &= \int_{X^n} f(x_1)\cdots f(x_n)E[D_{x_1}\cdots D_{x_n}F]\sigma(dx_1)\cdots\sigma(dx_n), \end{aligned}$$

hence the $\mathcal{S}$-transform satisfies

$$\begin{aligned} f \longmapsto \mathcal{S}F(f) : &= E\left[F \exp\left(-\int_X f(x)\sigma(dx)\right)\prod_{x\in X}(1+f(x))\right] \\ &= E[F\xi(f)] \\ &= \sum_{n=0}^{\infty}\frac{1}{n!}\int_{X^n} f(x_1)\cdots f(x_n)E[D_{x_1}\cdots D_{x_n}F]\sigma(dx_1)\cdots\sigma(dx_n). \end{aligned}$$

Addition operator

Consider the addition operator $\varepsilon_x^+ := I + D_x$, i.e.

$$\varepsilon_x^+ F(\omega) = F(\omega \cup \{x\}).$$

By a binomial transformation we find

$$g \longmapsto \mathcal{S}F(g) := \sum_{n=0}^{\infty} \frac{1}{n!} \int_{X^n} g(x_1) \cdots g(x_n) E[D_{x_1} \cdots D_{x_n} F] \sigma(dx_1) \cdots \sigma(dx_n)$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \int_{X^n} g(x_1) \cdots g(x_n) E[(\varepsilon_{x_1}^+ - I) \cdots (\varepsilon_{x_n}^+ - I) F] \sigma(dx_1) \cdots \sigma(dx_n)$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=0}^{n} (-1)^{n-k} \int_{X^n} g(x_1) \cdots g(x_n) \binom{n}{k} E[\varepsilon_{x_1}^+ \cdots \varepsilon_{x_k}^+ F] \sigma(dx_1) \cdots \sigma(dx_n)$$

$$= \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{n=k}^{\infty} (-1)^{n-k} \frac{1}{(n-k)!} \left(\int_X g(x) \sigma(dx) \right)^{n-k} \int_{X^k} g(x_1) \cdots g(x_k) E[\varepsilon_{x_1}^+ \cdots \varepsilon_{x_k}^+$$

$$= e^{-\int_X g(x) \sigma(dx)} \sum_{k=0}^{\infty} \frac{1}{k!} \int_{X^k} g(x_1) \cdots g(x_k) E[\varepsilon_{x_1}^+ \cdots \varepsilon_{x_k}^+ F] \sigma(dx_1) \cdots \sigma(dx_k).$$

$\mathcal{U}$-transform

Letting $g = e^f - 1$ we get the Laplace (or $\mathcal{U}$-) transform

$$E[Fe^{\int_X f d\omega}] = \sum_{n=0}^{\infty} \frac{1}{n!} E\left[F\left(\int_X f d\omega\right)^n\right] = e^{\int_X (e^f - 1) d\sigma} E[\xi(e^f - 1)F]$$

$$= \sum_{k=0}^{\infty} \frac{1}{k!} \int_{X^k} (e^{f(x_1)} - 1) \cdots (e^{f(x_k)} - 1) E\left[\varepsilon^+_{\mathfrak{x}_k} F\right] \sigma(dx_1) \cdots \sigma(dx_k)$$

$$= \sum_{k=0}^{\infty} \frac{1}{k!} \int_{X^k} \left(\sum_{n=1}^{\infty} \frac{f^n(x_1)}{n!}\right) \cdots \left(\sum_{n=1}^{\infty} \frac{f^n(x_k)}{n!}\right) E\left[\varepsilon^+_{\mathfrak{x}_k} F\right] \sigma(dx_1) \cdots \sigma(dx_k)$$

$$= \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{n=1}^{\infty} \sum_{d_1 + \cdots + d_k = n} \int_{X^k} \frac{f^{d_1}(x_1)}{d_1!} \cdots \frac{f^{d_k}(x_k)}{d_k!} E\left[\varepsilon^+_{\mathfrak{x}_k} F\right] \sigma(dx_1) \cdots \sigma(dx_k)$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=0}^{n} \frac{1}{k!} \sum_{d_1 + \cdots + d_k = n} \frac{n!}{d_1! \cdots d_k!} \int_{X^k} f^{d_1}(x_1) \cdots f^{d_k}(x_k) E\left[\varepsilon^+_{\mathfrak{x}_k} F\right] \sigma(dx_1) \cdots \sigma(d$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{P_1 \cup \cdots \cup P_k = \{1, \ldots, n\}} \int_{X^k} f^{|P_1|}(x_1) \cdots f^{|P_k|}(x_k) E\left[\varepsilon^+_{\mathfrak{x}_k} F\right] \sigma(dx_1) \cdots \sigma(dx_k).$$

Moments under a density

We find

$$E\left[F\left(\int_X f d\omega\right)^n\right] = \sum_{P_1\cup\cdots\cup P_k=\{1,\ldots,n\}} \int_{X^k} f^{|P_1|}(x_1)\cdots f^{|P_k|}(x_k) E\left[\varepsilon^+_{\mathfrak{x}_k} F\right] \sigma(dx_1)\cdots\sigma(dx_k).$$

When $Z = \omega(A) \simeq \mathcal{P}(\sigma(A))$ we have

$$E[FZ^n] = \sum_{k=0}^n S(n,k) \int_{A^k} E\left[\varepsilon^+_{\mathfrak{x}_k} F\right] \sigma(dx_1)\cdots\sigma(dx_k), \qquad n\in\mathbb{N},$$

where

$$S(n,k) = \frac{1}{k!} \sum_{d_1+\cdots+d_k=n} \frac{n!}{d_1!\cdots d_k!}$$

denotes the Stirling number of the second kind, i.e. the number of partitions of a set of n elements into k subsets. We also have

$$\begin{aligned} &E\left[F\int_X f_1 d\omega\cdots\int_X f_n d\omega\right] \\ &= \sum_{P_1\cup\cdots\cup P_k=\{1,\ldots,n\}} \int_{X^k} \prod_{i_1\in P_1} f_{i_1}(x_1)\cdots\prod_{i_k\in P_k} f_{i_k}(x_k) E\left[\varepsilon^+_{\mathfrak{x}_k} F\right] \sigma(dx_1)\cdots\sigma(dx_k). \end{aligned}$$

Moments of Poisson stochastic integrals

For u a *random* process we have

$$E\left[\left(\int_X u(x,\omega)\omega(dx)\right)^n\right] = \sum_{P_1\cup\cdots\cup P_k=\{1,\ldots,n\}} E\left[\int_{X^k} \varepsilon^+_{\mathfrak{x}_k}(u_{x_1}^{|P_1|}\cdots u_{x_k}^{|P_k|})\sigma(dx_1)\cdots\sigma(dx_k)\right].$$

In particular if f is deterministic,

$$E\left[\left(\int_X f(x)\omega(dx)\right)^n\right] = \sum_{P_1\cup\cdots\cup P_a=\{1,\ldots,n\}} \int_{X^a} f^{|P_1|}(x_1)\sigma(dx_1)\cdots\int_{X^a} f^{|P_a|}(x_a)\sigma(dx_a),$$

which follows from the relation

$$\begin{aligned} E[X^n] &= \sum_{k=0}^n \frac{1}{k!}\sum_{d_1+\cdots+d_k=n}\frac{n!}{d_1!\cdots d_k!}\kappa^X_{d_1}\cdots\kappa^X_{d_1} = \sum_{k=1}^n\sum_{B_1,\ldots,B_k}\kappa^X_{|B_1|}\cdots\kappa^X_{|B_k|} \\ &= A_n(\kappa^X_1,\kappa^X_2,\ldots,\kappa^X_n), \end{aligned}$$

between the moments and cumulants of a random variables, where

$$A_n(x_1,\ldots,x_n) = n!\sum_{\substack{r_1+2r_2+\cdots+nr_n=n\\ r_1,\ldots,r_n\geq 0}}\prod_{l=1}^n\left(\frac{1}{r_l!}\left(\frac{x_l}{l!}\right)^{r_l}\right)$$

is the Bell polynomial of degree n, cf. [BB90] for the differentiation of the Laplace transform of $\int_X f(x)\omega(dx)$.

For $n = 1$ we find the Mecke identity

$$E\left[\int_X u(x,\omega)\omega(dx)\right] = E\left[\int_{X^n} u_(x,\omega\cup\{x\})\sigma(dx)\right].$$

We have

$$E\left[F\left(\int_X u(x,\omega)\omega(dx)\right)^n\right] \quad = \quad \sum_{P_1,\dots,P_k} E\left[\int_{X^k} \varepsilon^+_{\mathfrak{x}_k}(Fu_{x_1}^{|P_1|}\cdots u_{x_k}^{|P_k|})\sigma(dx_1)\cdots\sigma(dx_k)\right]$$

Using the relation $\varepsilon^+_x = D_x + I$ we rewrite

$$\begin{aligned}
&\mathbb{E}\left[\left(\int_X u(x,\omega)\omega(dx)\right)^n\right]\\
&= \sum_{P_1\cup\cdots\cup P_k=\{1,\dots,n\}} \sum_{l=0}^{k} \binom{k}{l} \mathbb{E}\left[\int_{X^k} D_{s_1}\cdots D_{s_l}(u_{s_1}^{|P_1|}\cdots u_{s_k}^{|P_k|})\sigma(ds_1)\cdots\sigma(ds_k)\right]\\
&= \sum_{P_1\cup\cdots\cup P_k=\{1,\dots,n\}} \sum_{l=0}^{k} \binom{k}{l}\\
&\qquad \mathbb{E}\left[\int_{X^l} D_{s_1}\cdots D_{s_l}\left(u_{s_1}^{|P_1|}\cdots u_{s_l}^{|P_l|}\int_X u_{s_{l+1}}^{|P_1|}\sigma(ds_{l+1})\cdots\int_X u_{s_k}^{|P_k|}\sigma(ds_k)\right)\sigma(ds_1)\cdots\sigma(ds_l)\right].
\end{aligned}$$

Under the condition

$$D_{s_1} \cdots D_{s_k} (u_{s_1}^{|P_1|} \cdots u_{s_k}^{|P_k|}) = 0, \qquad k = 1, \ldots, n, \tag{2}$$

we find

$$\begin{aligned}
&\mathbb{E}\left[\left(\int_X u(x,\omega)\omega(dx)\right)^n\right] \\
&= \sum_{P_1 \cup \cdots \cup P_k = \{1,\ldots,n\}} \sum_{l=0}^{k-1} \binom{k}{l} \mathbb{E}\left[\int_{X^k} D_{s_1} \cdots D_{s_l} (u_{s_1}^{|P_1|} \cdots u_{s_k}^{|P_k|}) \sigma(ds_1) \cdots \sigma(ds_k)\right] \\
&= \sum_{P_1 \cup \cdots \cup P_k = \{1,\ldots,n\}} \sum_{l=0}^{k-1} \binom{k}{l} \\
&\quad \times \mathbb{E}\left[\int_{X^{k-1}} D_{s_1} \cdots D_{s_l} \left(u_{s_1}^{|P_1|} \cdots u_{s_{k-1}}^{|P_{k-1}|} \int_X u_s^{|P_k|} \sigma(ds)\right) \sigma(ds_1) \cdots \sigma(ds_{k-1})\right].
\end{aligned}$$

If, in addition to

$$D_{s_1} \cdots D_{s_k}(u_{s_1}^{|P_1|} \cdots u_{s_k}^{|P_k|}) = 0, \qquad k = 1, \ldots, n,$$

the integral $\int_X u_s^k \sigma(ds)$ is deterministic for all $k = 1, \ldots, n$, then by a decreasing induction on k we show that

$$\begin{aligned} \mathbb{E}\left[\left(\int_X u(x,\omega)\omega(dx)\right)^n\right] &= \sum_{P_1 \cup \cdots \cup P_k = \{1,\ldots,n\}} \int_{X^k} u_{s_1}^{|P_1|} \cdots u_{s_k}^{|P_k|} \sigma(ds_1) \cdots \sigma(ds_k) \\ &= \sum_{P_1 \cup \cdots \cup P_k = \{1,\ldots,n\}} \int_X u_{s_1}^{|P_1|} \sigma(ds_1) \cdots \int_X u_{s_k}^{|P_k|} \sigma(ds_k), \end{aligned}$$

i.e. $\displaystyle\int_X u(x,\omega)\omega(dx)$ has an infinitely divisible distribution with cumulants $\int_X u_s^k \sigma(ds)$, $k \geq 1$.

Cyclic condition

Proposition

Let $n \geq 1$ and $\omega \in \Omega^X$ and assume that the process $u : \Omega \times X \longrightarrow \mathbb{R}$ satisfies the cyclic condition

$$D_{x_1} u(x_2, \omega) \cdots D_{x_{k-1}} u(x_k, \omega) D_{x_k} u(x_1, \omega) = 0, \tag{3}$$

for all $x_1, \ldots, x_k \in X$, $k = 1, \ldots, n$. Then we have

$$D_{x_1} \cdots D_{x_k} (u(x_1, \omega) \cdots u(x_k, \omega)) = 0,$$

for all $x_1, \ldots, x_k \in X$, $k = 1, \ldots, n$.

The cyclic condition (3) holds if there exists $i \in \{1, \ldots, k\}$ such that

$$D_{x_i} u(x_{i+1 \mod k}, \omega) = 0,$$

for all $\omega \in \Omega$.

Invariance of Poisson measures

Given a measurable random process

$$\tau : \Omega^X \times X \to Y,$$

indexed by X, $\tau_*(\omega)$, $\omega \in \Omega^X$, maps

$$\omega = \sum_{x \in \omega} \epsilon_x \in \Omega^X \quad \text{to} \quad \tau_*(\omega) = \sum_{x \in \Omega} \epsilon_{\tau(x)} \in \Omega^Y,$$

i.e. $\tau_* : \Omega^X \to \Omega^Y$ shifts each configuration point $x \in \omega$ according to $x \mapsto \tau(\omega, x)$.
Assume that $\tau : \Omega^X \times X \to Y$ map σ to μ, i.e.

$$\tau_*(\omega, \cdot)\sigma = \mu, \qquad \omega \in \Omega^X,$$

and satisfies the cyclic condition

$$D_{t_1}\tau(\omega, t_2) \cdots D_{t_k}\tau(\omega, t_1) = 0,$$

$t_1, \ldots, t_k \in X$, $\omega \in \Omega^X$, $k \geq 1$. Then $\tau_* : \Omega^X \to \Omega^Y$ maps $\mathbb{P}_\sigma$ to $\mathbb{P}_\mu$, i.e.

$$\tau_* \mathbb{P}_\sigma = \mathbb{P}_\mu$$

is the Poisson measure with intensity $\mu(dy)$ on Y.

Proof.
Apply

$$\begin{aligned}\mathbb{E}\left[\left(\int_X u(x,\omega)\omega(dx)\right)^n\right] &= \sum_{P_1\cup\cdots\cup P_k=\{1,\ldots,n\}} \int_{X^k} u_{s_1}^{|P_1|}\cdots u_{s_k}^{|P_k|}\sigma(ds_1)\cdots\sigma(ds_k) \\ &= \sum_{P_1\cup\cdots\cup P_k=\{1,\ldots,n\}} \int_X u_{s_1}^{|P_1|}\sigma(ds_1)\cdots\int_X u_{s_k}^{|P_k|}\sigma(ds_k),\end{aligned}$$

to $u(x,\omega)=h(\tau(x,\omega))$ where h is a deterministic function. □

When Y is a metric space the cyclic condition

$$D_{t_1}\tau(\omega,t_2)\cdots D_{t_k}\tau(\omega,t_1)=0,$$

is interpreted by saying that for all $k\geq 1$ there is $l\in\{1,\ldots,k\}$ such that

$$\tau(\omega\cup\{t_l\},t_{l+1}),=\tau(\omega,t_{l+1}), \qquad l=1,\ldots,k,$$

with $k+1\simeq 1$.

The cyclic condition is an extension of the usual adaptedness condition, which holds when $\tau : X\to X$ is adapted to a given total binary relation $\prec$ on X.

Quasi-invariance and Girsanov identities

We wish to extend

$$E\left[I_k(f^{\otimes k})F\right] = E\left[\int_{X^k} D_{\mathfrak{s}_k}(Ff(s_1)\cdots f(s_k))\right]\sigma(ds_1)\cdots\sigma(ds_k), \quad (4)$$

to a random process u, i.e.

$$E[I_k(u^{\otimes k})] = E\left[\int_{X^k} D_{s_1}\cdots D_{s_k}(u_{s_1}\cdots u_{s_k})ds_1\cdots ds_k\right]$$

for any simple process u of the form

$$u = \sum_{i=1}^{n} F_i f_i.$$

We may apply the above identity towards an extension to indicator functions over random sets by considering pathwise extensions of the multiple stochastic integral. However this can lead to technical difficulties and for this reason we prefer to apply Stirling inversion.

Stopping sets

For $A(\omega)$ a random set and F a bounded random variable we have

$$E\left[FI_n(\mathbf{1}_A^{\otimes n})\right] = E_\sigma\left[\int_{X^n} D_{s_1}\cdots D_{s_n}\left(F\prod_{p=1}^n \mathbf{1}_A(s_p)\right)\sigma(ds_1)\cdots\sigma(ds_n)\right].$$

We note that

$$I_n(\mathbf{1}_A^{\otimes n}) = C_n(\omega(A), \sigma(A))$$

where $C_n(x, \lambda)$ is the Charlier polynomial

$$C_n(\omega(A), \sigma(A)) = \sum_{l=0}^n \sum_{k=0}^n (\omega(A))^k \binom{n}{l}(-\sigma(A))^{n-l} s(l,k).$$

Recall that

$$E[F(\omega(A))^n] = \sum_{k=0}^n S(n,k)\int_X E\left[\varepsilon^+_{\mathfrak{x}_k}(F\mathbf{1}_{A(\omega)}(x_1)\cdots\mathbf{1}_{A(\omega)}(x_k))\right]\sigma(dx_1)\cdots\sigma(dx_k), \qquad n\in\mathbb{N}.$$

Proof

Stirling inversion yields

$$
\begin{aligned}
E\left[FI_n(\mathbf{1}_A^{\otimes n})\right] &= E\left[FC_n(\omega(A),\sigma(A))\right] \\
&= \sum_{l=0}^{n}\sum_{k=0}^{n} E\left[F\left(\omega(A)\right)^k \binom{n}{l}(-\sigma(A))^{n-l}s(l,k)\right] \\
&= \sum_{l=0}^{n}\binom{n}{l}s(l,k)\sum_{k=0}^{n}\sum_{j=0}^{k}S(k,j)E\left[\int_{X^j}\varepsilon_{\mathfrak{s}_j}^{+}(F(-\sigma(A))^{n-l}\mathbf{1}_{A(\omega)}(s_1)\cdots\mathbf{1}_{A(\omega)}(s_j))\sigma(d\right. \\
&= \sum_{l=0}^{n}(-1)^{n-l}\binom{n}{l}\sum_{j=0}^{n}\sum_{k=0}^{n}s(l,k)S(k,j) \\
&\quad \times E\left[\int_{X^{n-l+j}}\varepsilon_{\mathfrak{s}_j}^{+}(F\mathbf{1}_{A(\omega)}(s_1)\cdots\mathbf{1}_{A(\omega)}(s_{n-l+j}))\sigma(ds_1)\cdots\sigma(ds_{n-l+j})\right] \\
&= \sum_{l=0}^{n}(-1)^{n-l}\binom{n}{l}E\left[\int_{X^n}\varepsilon_{\mathfrak{s}_l}^{+}(F\mathbf{1}_{A(\omega)}(s_1)\cdots\mathbf{1}_{A(\omega)}(s_n))\sigma(ds_1)\cdots\sigma(ds_n)\right] \\
&= E_\sigma\left[\int_{X^n}D_{s_1}\cdots D_{s_n}\left(F\prod_{p=1}^{n}\mathbf{1}_A(s_p)\right)\sigma(ds_1)\cdots\sigma(ds_n)\right].
\end{aligned}
$$

Stopping sets

Assume that the random set $A(\omega)$ satisfies the cyclic condition

$$D_{t_1}\mathbf{1}_A(t_2)\cdots D_{t_k}\mathbf{1}_A(t_1)=0, \qquad \sigma(dt_1),\ldots,\sigma(dt_k)-a.e., \quad \omega\in\Omega^X, \tag{5}$$

for all $k\geq 1$ and that F is such that

$$\mathbf{1}_A(x)D_xF=0, \qquad x\in X,$$

$x\notin\{x_1,\ldots,x_k\}$. Then we have

$$E\left[FI_n(\mathbf{1}_A^{\otimes n})\right]=0, \qquad n\geq 1.$$

Stopping sets

We have

$$\begin{aligned} & D_{x_1} \cdots D_{x_k}(Fu(x_1,\omega)\cdots u(x_k,\omega)) \\ & = \sum_{\Theta_1\cup\cdots\cup\Theta_k=\{1,\ldots,k\}} D_{\Theta_1}(Fu(x_1,\omega))\cdots D_{\Theta_k}u(x_k,\omega), \end{aligned} \tag{6}$$

where the above sum runs over possibly empty subsets $\Theta_1,\ldots,\Theta_k$. This quantity is known to vanish under the cyclic condition

$$D_{x_1}(Fu(x_2,\omega))\cdots D_{x_k}u(x_1,\omega) = 0,$$

$x_1,\ldots,x_k \in X$, $\omega \in \Omega^X$, $k \geq 1$. We have

$$\begin{aligned} & D_{x_1}Fu(x_2,\omega))\cdots D_{x_k}u(x_1,\omega) \\ = \quad & FD_{x_1}u(x_2,\omega)\cdots D_{x_k}u(x_1,\omega) \\ & +D_{x_1}FD_{x_1}u(x_2,\omega)\cdots D_{x_k}u(x_1,\omega) \\ & +u(x_2,\omega)D_{x_1}FD_{x_2}u(x_3,\omega)\cdots D_{x_k}u(x_1,\omega), \end{aligned}$$

$\sigma(dx_1),\ldots,\sigma(dx_k)-a.e.$, $\omega \in \Omega^X$, for all $k \geq 1$.

Stopping sets

Proposition

Assume that $\phi : \Omega^X \times X \to \mathbb{R}_+$ *is a nonnegative process that satisfies the cyclic condition*

$$D_{t_1}\phi(\omega, t_2) \cdots D_{t_k}\phi(\omega, t_1) = 0, \qquad \sigma(dt_1), \ldots, \sigma(dt_k) - a.e., \quad \omega \in \Omega^X, \tag{7}$$

for all $k \geq 1$, $\omega \in \Omega^X$, $x \in X$, *with*

$$E_\sigma\left[Fe^{\int_X \phi(\omega,x)\sigma(dx)} \prod_{x\in\omega}(1+\phi(\omega,x))\right] < \infty,$$ *where* F *is such that*

$$\phi(\omega, t)D_t F = 0, \qquad t \in X.$$

Then we have

$$E_\sigma\left[Fe^{-\int_X \phi(\omega,x)\sigma(dx)} \prod_{x\in\omega}(1+\phi(\omega,x))\right] = E_\sigma[F].$$

Stopping sets

Given $K \in \mathcal{B}(X)$, let

$$\mathcal{F}_K = \sigma(\omega(U) \ : \ U \subset K)$$

denote the σ-algebra generated by $\omega(U)$, $U \subset K$. We refer to [Zuy99] and Def. 2.27 in [Mol05] for the notion of stopping set.

Definition

A stopping set is a random set $A(\omega)$ such that

$$\{A \subset U\} \in \mathcal{F}_K \quad \text{for all} \quad U \subset K.$$

Lemma

For A a stopping set we have

$$D_t \mathbf{1}_{A(\omega)}(s) = 0, \qquad s \in X, \quad t \in A(\omega)^c, \quad \omega \in \Omega.$$

Stopping sets

Examples of stopping sets.

1. When $X = \mathbb{R}_+$, the interval $[0, \tau]$, where τ is a stopping time, is a stopping set.
2. All intervals $[0, T_n]$, where T_n is the n-th Poisson jump time, are stopping sets.
3. When $X = \mathbb{R}_+$, the interval $[T_{N_T}, T]$ is a decreasing stopping set - note that $\mathbf{1}_{[0, T_{N_T})}(t)$ is predictable with respect to the *backward* Poisson filtration.
4. The minimal closed ball centered in the origin and containing exactly n points is a stopping set.
5. The Poisson-Voronoi flower is a stopping set.
6. The complement of a convex Poisson hull inside a convex set is a decreasing stopping set.

Stopping sets

Proposition

Let A be a $\mathcal{C}$ a decreasing *stopping set, i.e.*

$$A(\omega \cup \{x\}) \subset A(\omega), \qquad x \in X.$$

Then $\mathbf{1}_A$ satisfies the cyclic Condition (7)*, i.e.*

$$D_{t_1}\mathbf{1}_A(t_2)\cdots D_{t_k}\mathbf{1}_A(t_1) = 0, \qquad \sigma(dt_1),\ldots,\sigma(dt_k) - a.e., \quad \omega \in \Omega^X,$$

for all $k \geq 2$,

Example: the case of T_n

Take

$$u_t = z\mathbf{1}_{[0,T_n]}(t), \qquad t \in \mathbb{R}_+.$$

We find

$$E[e^{-zT_n}(1+z)^n] = 1,$$

hence

$$E[e^{-zT_n}] = (1+z)^{-n},$$

and $[0, T_n]$ is stopping set.

Example: the case of T_{N_T}

The random time T_{N_T} is not a stopping time, however T_{N_T} can be seen as the first jump time of a Poisson process ran backward and the process

$$u(t,\omega) = z\mathbf{1}_{[T_{N_T},T]}(t), \qquad t \in \mathbb{R}_+,$$

is backward predictable on $[0, T]$. We get

$$E\left[e^{-z(T-T_{N_T})}(1+z)^{N_T - N_{T_{N_T}}}\right] = 1,$$

hence

$$E\left[e^{-z(T-T_{N_T})}\mathbf{1}_{\{T_1<T\}}\right] = \frac{1-e^{-(1+z)T}}{1+z}.$$

Example: the case of T_{N_T}

The random time T_{N_T} is not a stopping time, however $[T_{N_T}, T]$ is a decreasing stopping set hence we also get

$$E\left[e^{-z(T-T_{N_T})}(1+z)^{N_T-N_{T_{N_T}}}\right]=1.$$

Remark: we may also use

$$E\left[f(T_{T_{N_T}})e^{-zT_{N_T}}(1+z)^{N_{T_{N_T}}-1}\right]=E[f(T_{T_{N_T}})],$$

obtained from

$$E\left[F\xi(1_A)\right]=E[F]$$

with $A=[0, T_{N_T})$.

Example: the case of T_{N_T}

Indeed,

$$
\begin{aligned}
& E\left[f(T_{N_T})e^{-zT_{N_T}}(1+z)^{N_T-1}\mathbf{1}_{\{N_T\geq 1\}} + f(0)\mathbf{1}_{\{N_T=0\}}\right] \\
&= e^{-T}f(0) + \sum_{n=1}^{\infty}\frac{(1+z)^{n-1}}{(n-1)!}\int_0^T e^{-(T-y)}f(y)e^{-zy}y^{n-1}e^{-y}dy \\
&= e^{-T}f(0) + e^{-T}\sum_{n=1}^{\infty}\frac{(1+z)^{n-1}}{(n-1)!}\int_0^T f(y)e^{-zy}y^{n-1}dy \\
&= e^{-T}f(0) + e^{-T}(1+z)^{n-1}\int_0^T f(y)e^{-zy}\sum_{n=1}^{\infty}\frac{y^{n-1}}{(n-1)!}dy \\
&= e^{-T}f(0) + e^{-T}\int_0^T f(y)e^{-zy}e^{y(1+z)}dy \\
&= e^{-T}f(0) + e^{-T}\int_0^T f(y)e^{y}dy \\
&= E[f(T_{N_T})],
\end{aligned}
$$

Example: the case of T_{N_T}

Now, from

$$E[f(T_{N_T})] = E\left[f(T_{N_T})e^{-zT_{N_T}}(1+z)^{N_T-1}\right],$$

we can take

$$f(x) = e^{xz}\mathbf{1}_{(0,\infty)}(x),$$

to get

$$\begin{aligned} E[(1+z)^{N_{T_{N_T}}-1}\mathbf{1}_{\{N_T\geq 1\}}] &= E[f(T_{N_T})e^{-zT_{N_T}}(1+z)^{N_{T_{N_T}}-1}] \\ &= E\left[f(T_{N_T})e^{-zT_{N_T}}(1+z)^{N_T-1}\mathbf{1}_{\{N_T\geq 1\}}\right] \\ &= E[f(T_{N_T})] \\ &= E\left[e^{zT_{N_T}}\mathbf{1}_{\{N_T\geq 1\}}\right], \end{aligned}$$

which yields

$$\begin{aligned} E[e^{zT_{N_T}}\mathbf{1}_{\{N_T\geq 1\}}] &= E\left[(1+z)^{N_T-1}\mathbf{1}_{\{N_T\geq 1\}}\right] \\ &= \frac{1}{1+z}\left(\frac{e^{zT}-e^{-T}}{1-e^{-T}}\right), \qquad z > -1, \end{aligned}$$

Laplace transform of the truncated exponential distribution.

Stopping sets

Given S stopping set, let

$$u_x = z\mathbf{1}_{X \setminus S}(t), \qquad t \in \mathbb{R}_+,$$

and

$$F = \exp(z\sigma(X \setminus S)),$$

we find

$$E[e^{z(\sigma(X \setminus S))}] = E[(1+z)^{\omega(X \setminus S)}].$$

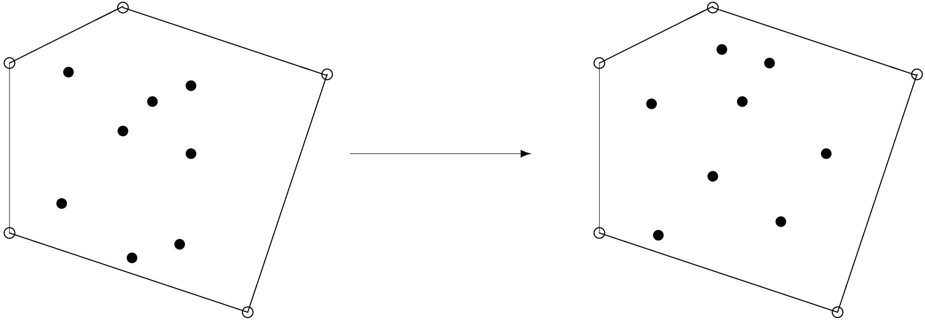
Similarly, letting

$$F = e^{z(\sigma(X) - \sigma(S))}\mathbf{1}_{\{\omega(S)=n\}},$$

we find

$$\begin{aligned} E[e^{-z\sigma(S)}\mathbf{1}_{\{\omega(S)=n\}}] &= e^{-z\sigma(X)}E[(1+z)^{\omega(X)-\omega(S)}\mathbf{1}_{\{\omega(S)=n\}}] \\ &= (1+z)^{-n}e^{-z\sigma(X)}E[(1+z)^{\omega(X)}\mathbf{1}_{\{\omega(S)=n\}}] \\ &= (1+z)^{-n}E_z[\mathbf{1}_{\{\omega(S)=n\}}]. \end{aligned}$$

Example: convex hull



Consider the order

$$x \preceq y \quad \iff x \in \mathcal{C}(\omega \cup \{y\}),$$

for example $\tau(\omega, \cdot) : X \to X$ modifies only the inside points of the convex hull of ω, depending on the positions of the extremal vertices.

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