Joint asymptotic distribution of certain path-functionals of the reflected process

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Joint work with A. Mijatovic

Motivation

The reflected process & applicantions

- The reflected process Y of a Lévy process X: strong Markov process equal to X reflected at its running infimum

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The reflected process & applicantions

- The reflected process Y of a Lévy process X: strong Markov process equal to X reflected at its running infimum
- The reflected process plays a role in various applications:
  - method of cumulative sums (CUSUM) used in **mathematical statistics** (e.g. Moustakides (2004), Shiryaev (1996)),
  - **queueing theory and risk theory** (e.g. Iglehart (1972), Asmussen (1982), Asmussen (2003)),
  - **mathematical finance**: draw-down, Russian option (e.g. Hadjiliadis and Vecer (2006), Avram, Kyprianou, P (2004))
  - **mathematical genetics** (e.g. Karlin and Brendel (Science, 1992)).

Motivation

Example from mathematical genetics

Table 2. Scores for DNA-binding domains.

Residue	q^*	$p\dagger$	$\log_2(q/p)$	Score‡
C	0.033	0.015	1.099	4
R	0.118	0.062	0.940	4
K	0.096	0.057	0.757	3
W	0.015	0.010	0.677	3
Y	0.034	0.027	0.316	1
F	0.037	0.032	0.215	1
I	0.051	0.044	0.201	1
N	0.045	0.043	0.063	0
Q	0.053	0.053	0.024	0
E	0.067	0.067	0.010	0
T	0.054	0.054	−0.020	0
V	0.049	0.053	−0.095	0
L	0.080	0.091	−0.184	−1
H	0.022	0.025	−0.200	−1
A	0.067	0.079	−0.241	−1
M	0.019	0.024	−0.376	−2
G	0.047	0.064	−0.454	−2
S	0.057	0.087	−0.615	−2
D	0.029	0.050	−0.799	−3
P	0.027	0.062	−1.223	−5

*Frequencies in the aggregate of 753 annotated DNA-binding domains assembled from SWISS-PROT Release 21.0 (*25*). †Average overall frequencies in the same set of DNA-binding proteins. ‡Values of the previous column multiplied by the scale factor 4 and rounded to the nearest integer.

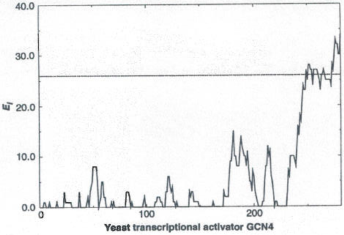

Fig. 2. Identification of high-scoring DNA-binding segments in the yeast transcriptional activator GCN4 (*2*). Scoring assignments were as given in Table 2. The dotted line indicates the significance threshold at the 5% level calculated according to Eq. 1. The high-scoring COOH-terminal sequence contains the DNA-binding function of GCN4 (*2*).

Setting
Definitions

Fluctuations of Y are described by among others

- $\tau(x) \doteq \inf\{t \geq 0 : Y(t) \in (x, \infty)\}$
- $Y^*(t) \doteq \sup_{0 \leq s \leq t} Y(s)$

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- $Y^*(t) \doteq \sup_{0 \leq s \leq t} Y(s)$
- **Aim of this talk:** describe the **asymptotic dependence** and **weak limiting behaviour** of the following functionals of the reflected process Y:

$$Y(t) \doteq X(t) - \inf_{0 \leq s \leq t} X(s),$$

$$M(t, x) \doteq Y^*(t) - x,$$

$$Z(x) \doteq Y(\tau(x)) - x.$$

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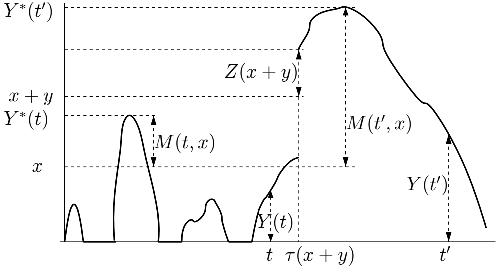

FIGURE 1. This schematic figure of a path of Y depicts the values of the three functionals at times t and t', before and after the reflected process crosses the level $x+y$. It is intuitively clear that, in general, $M(t,x)$, $Z(x+y)$ and $Y(t)$ cannot be independent for fixed $t,x,y>0$.

Setting

Cramér setting

- **Standing assumption.**
  The mean of $X(1)$ is finite, *Cramér's condition*, $E[\mathrm{e}^{\gamma X(1)}] = 1$ for $\gamma > 0$, holds, $E[\mathrm{e}^{\gamma X(1)}|X(1)|] < \infty$ and the Lévy measure of X is non-lattice.

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- **Implications**:
  - $E[X(1)]$ is strictly negative
  - Lévy measure ν satisfies

$$\int_1^\infty \mathrm{e}^{\gamma x}\nu(\mathrm{d}x) < \infty.$$

Asymptotic distributions

Main result

- **Definition.** A family of random vectors $\{(U_z^1, \ldots, U_z^d)\}_{z \in \mathbb{R}_+^l}$ on a given probability space, where $d, l \in \mathbb{N}$, is *asymptotically independent* if the joint CDF is asymptotically equal to a product of the CDFs of the components: i.e. for any $a_i \in (-\infty, \infty]$, $i = 1, \ldots, d$, it holds

$$P(U_z^1 \le a_1, \ldots, U_z^d \le a_d) = \prod_{i=1}^{d} P(U_z^i \le a_i) + o(1), \text{ as } \min\{z_i\} \to \infty$$

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- **Notation:** ϕ: the Laplace exponent of the increasing ladder-height process H of X
- **Theorem.**
  The triplet $\{(Y(t), Z(x+y), M(t,x))\}_{t,x,y \in \mathbb{R}_+}$ is asymptotically independent and the weak limit $Z(x) \xrightarrow{\mathcal{D}} Z(\infty)$, as $x \to \infty$, exists, where

$$E[\mathrm{e}^{-vZ(\infty)}] = \frac{\gamma}{\gamma + v} \cdot \frac{\phi(v)}{\phi(0)} \qquad \text{for all} \quad v \in \mathbb{R}_+.$$

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Remarks.

- In general, the functionals $M(t,x)$ and $Z(x)$ are not asymptotically independent as $t \wedge x \to \infty$. However, the Theorem implies that for any $\alpha > 1$ the variables $M(t,x)$ and $Z(\alpha x)$ are asymptotically independent as $t \wedge x \to \infty$.
- If the Lévy measure of X is heavy-tailed, the "single-large jump"-heuristic suggests that $Z(x+y)$ and $M(t,x)$ are not asymptotically independent.

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What about the marginal asymptotic distributions of $Y(t)$ and $M(t,x)$?

- The Cramér's condition implies that X tends to $-\infty$ almost surely, and hence, by the classical time reversal argument, the reflected process Y has a stationary distribution $Y(\infty)$ equal to the law of the ultimate supremum $\sup_{t\geq 0} X(t)$.

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- The Cramér's condition implies that X tends to $-\infty$ almost surely, and hence, by the classical time reversal argument, the reflected process Y has a stationary distribution $Y(\infty)$ equal to the law of the ultimate supremum $\sup_{t\geq 0} X(t)$.
- Doney & Maller (2005)'s Cramér esimate under the excursion measure implies that $M(t,x)$ converges weakly to a Gumbel distribution if the quantity $t\mathrm{e}^{-\gamma x}$ tends to a positive constant as $t, x \uparrow \infty$

Asymptotic distributions

- **Corollary 1.** Let $m \doteq \lim_{u\to\infty} \phi(u)/u$, H upcrossing ladder height process, and $\overline{\nu}_H(x) \doteq \nu_H((x,\infty))$, $x > 0$. Then the law of the asymptotic overshoot $Z(\infty)$ is given by:

$$P(Z(\infty) > x) = \frac{\gamma}{\phi(0)} \mathrm{e}^{-\gamma x} \int_x^\infty \mathrm{e}^{\gamma y}\, \overline{\nu}_H(y)\, \mathrm{d}y, \quad x \in [0,\infty).$$

In particular, $Z(\infty)$ is a continuous random variable except possibly at the origin.

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- **Corollary 2.** The weak limit of the random vector $(Y(t), Z(x))$, as $x \wedge t \to \infty$, exists and the law $(Y(\infty), Z(\infty))$ is determined by the joint Laplace transform

$$E[\exp(-uY(\infty) - vZ(\infty))] = \frac{\gamma}{\gamma + v} \cdot \frac{\phi(v)}{\phi(u)} \qquad \text{for all} \quad u, v \in \mathbb{R}_+.$$

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- **Corollary 3.** Assume that $\lim_{t\uparrow\infty} t\mathrm{e}^{-\gamma x} = \lambda$ for some $\lambda > 0$. Then $(Y(t), Z(x+y), M(t,x))$ converges weakly as $t \wedge y \to \infty$ and the joint limit law $(Y(\infty), Z(\infty), M(\infty))$ is given by the Fourier-Laplace transform:

$$E\left[\exp\left(-uY(\infty) - vZ(\infty) + \mathrm{i}\beta M(\infty)\right)\right] = \frac{\gamma}{\gamma + v} \cdot \frac{\phi(v)}{\phi(u)}$$
$$\cdot\Gamma\left(1 - \frac{\mathrm{i}\beta}{\gamma}\right) \cdot \exp\left[\mathrm{i}\beta\gamma^{-1}\log\left(\lambda\ell C_\gamma\widehat{\phi}(\gamma)\right)\right]$$

for all $u, v \in \mathbb{R}_+$, $\beta \in \mathbb{R}$, where $\ell \doteq 1/E[\widehat{L}^{-1}(1)]$ and $C_\gamma \doteq \frac{\phi(0)}{\gamma\phi'(-\gamma)}$.

Sketch of the proof: asymptotic independence

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- **Splitting property of a Poisson point process** η at first entrance into A:

$$\eta^A := \{\eta_t : t < H_A\} \perp \eta(H_A)$$

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- In fact, as in Greenwood & Pitman (1980) we will also consider the excursion process $\epsilon' = (\epsilon, \tilde{\epsilon})$ of (Y, N^q) away from $\{0\} \times \mathbb{R}_+$ where N^q is an independent Poisson process with rate q.

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- For example, splitting at $H_{B'}$ with $B' = \{\epsilon' : \rho(x,\epsilon) < \zeta(\epsilon)\}$ and noting $\widehat{L}(\tau(x)) = H_{B'}$ yields

$$P(\widehat{L}(\tau(x)) > \widehat{L}(e_q), Y(e_q) \in A, Z(x) \in B)$$
$$= P(\widehat{L}(\tau(x)) > \widehat{L}(e_q), Y(e_q) \in A) P(Z(x) \in B),$$

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- The proof is based on such splitting arguments, inversion of the Laplace transforms and estimates on such as the following:
  Lemma. The following limits hold, as $x \wedge t \to \infty$:

$$P(\widehat{L}(t) = \widehat{L}(\tau(x))) \longrightarrow 0$$

$$\limsup_{x \wedge t \to \infty} P(\widehat{L}(t(1-\delta_1)) \leq \widehat{L}(\tau(x)) \leq \widehat{L}(t(1-\delta_2))) \leq \frac{8(\delta_1 \vee \delta_2)}{\mathrm{e}}$$

$$(\forall \delta_1, \delta_2 \in [0, 1/4))$$

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$$P(\epsilon(H_A) \in B) = n(B|A) := n(B)/n(A), \qquad B \subset A, n(A) > 0,$$

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- Taking

  $$A = \{\epsilon : \rho(x,\epsilon) < \zeta(\epsilon)\}, \;\; B = \{\epsilon : \epsilon(\rho(x,\epsilon)) - x > y, \rho(x,\epsilon) < \zeta(\epsilon)\}$$

  with $\rho(x,\epsilon) = \inf\{t \geq 0 : \epsilon(t) > x\}$, we have

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  with $\rho(x,\epsilon) = \inf\{t \geq 0 : \epsilon(t) > x\}$, we have

  $$P(Z(x) > y) = n(\epsilon(\rho(x,\epsilon))) - x > y|\rho(x,\epsilon) < \zeta(\epsilon)).$$
- **Lemma.** As $x \to \infty$, we have

  $$n(\mathrm{e}^{-v(\epsilon(\rho(x))-x)}|\rho(x) < \zeta) \to \frac{\gamma}{\gamma + v} \cdot \frac{\phi(v)}{\phi(0)}.$$