

Joint asymptotic distribution
of certain path-functionals
of the reflected process

Martijn Pistorius

Imperial College London/Univ. of Amsterdam

**7th International Conference on Lévy processes
Theory and Applications
Wroclaw, Poland, July 16, 2013**

Joint work with A. Mijatovic

Motivation

The reflected process & applications

- ▶ The reflected process Y of a Lévy process X : strong Markov process equal to X reflected at its running infimum

Motivation

The reflected process & applications

- ▶ The reflected process Y of a Lévy process X : strong Markov process equal to X reflected at its running infimum
- ▶ The reflected process plays a role in various applications:
 - ▶ method of cumulative sums (CUSUM) used in **mathematical statistics** (e.g. Moustakides (2004), Shiryaev (1996)),
 - ▶ **queueing theory and risk theory** (e.g. Iglehart (1972), Asmussen (1982), Asmussen (2003)),
 - ▶ **mathematical finance**: draw-down, Russian option (e.g. Hadjilias and Vecer (2006), Avram, Kyprianou, P (2004))
 - ▶ **mathematical genetics** (e.g. Karlin and Brendel (Science, 1992)).

Motivation

Example from mathematical genetics

Table 2. Scores for DNA-binding domains.

Residue	q^*	pt	$\log_2(q/p)$	Score [‡]
C	0.033	0.015	1.099	4
R	0.118	0.062	0.940	4
K	0.096	0.057	0.757	3
W	0.015	0.010	0.677	3
Y	0.034	0.027	0.316	1
F	0.037	0.032	0.215	1
I	0.051	0.044	0.201	1
N	0.045	0.043	0.063	0
Q	0.053	0.053	0.024	0
E	0.067	0.067	0.010	0
T	0.054	0.054	-0.020	0
V	0.049	0.053	-0.095	0
L	0.080	0.091	-0.184	-1
H	0.022	0.025	-0.200	-1
A	0.067	0.079	-0.241	-1
M	0.019	0.024	-0.376	-2
G	0.047	0.064	-0.454	-2
S	0.057	0.087	-0.615	-2
D	0.029	0.050	-0.799	-3
P	0.027	0.062	-1.223	-5

*Frequencies in the aggregate of 753 annotated DNA-binding domains assembled from SWISS-PROT Release 21.0 (25). †Average overall frequencies in the same set of DNA-binding proteins. ‡Values of the previous column multiplied by the scale factor 4 and rounded to the nearest integer.

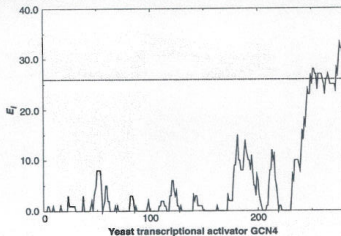


Fig. 2. Identification of high-scoring DNA-binding segments in the yeast transcriptional activator GCN4 (2). Scoring assignments were as given in Table 2. The dotted line indicates the significance threshold at the 5% level calculated according to Eq. 1. The high-scoring COOH-terminal sequence contains the DNA-binding function of GCN4 (2).

Setting

Definitions

Fluctuations of Y are described by among others

- ▶ $\tau(x) \doteq \inf\{t \geq 0 : Y(t) \in (x, \infty)\}$
- ▶ $Y^*(t) \doteq \sup_{0 \leq s \leq t} Y(s)$

Setting

Definitions

Fluctuations of Y are described by among others

- ▶ $\tau(x) \doteq \inf\{t \geq 0 : Y(t) \in (x, \infty)\}$
- ▶ $Y^*(t) \doteq \sup_{0 \leq s \leq t} Y(s)$
- ▶ **Aim of this talk:** describe the **asymptotic dependence** and **weak limiting behaviour** of the following functionals of the reflected process Y :

$$Y(t) \doteq X(t) - \inf_{0 \leq s \leq t} X(s),$$

$$M(t, x) \doteq Y^*(t) - x,$$

$$Z(x) \doteq Y(\tau(x)) - x.$$

Setting

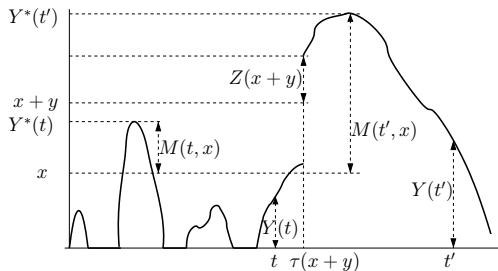


FIGURE 1. This schematic figure of a path of Y depicts the values of the three functionals at times t and t' , before and after the reflected process crosses the level $x+y$. It is intuitively clear that, in general, $M(t, x)$, $Z(x+y)$ and $Y(t')$ cannot be independent for fixed $t, x, y > 0$.

Setting

Cramér setting

► **Standing assumption.**

The mean of $X(1)$ is finite, *Cramér's condition*, $E[e^{\gamma X(1)}] = 1$ for $\gamma > 0$, holds, $E[e^{\gamma X(1)} | X(1)] < \infty$ and the Lévy measure of X is non-lattice.

Setting

Cramér setting

- ▶ **Standing assumption.**

The mean of $X(1)$ is finite, *Cramér's condition*, $E[e^{\gamma X(1)}] = 1$ for $\gamma > 0$, holds, $E[e^{\gamma X(1)} | X(1)] < \infty$ and the Lévy measure of X is non-lattice.

- ▶ **Implications:**

- ▶ $E[X(1)]$ is strictly negative
- ▶ Lévy measure ν satisfies

$$\int_1^{\infty} e^{\gamma x} \nu(dx) < \infty.$$

Asymptotic distributions

Main result

- ▶ **Definition.** A family of random vectors $\{(U_z^1, \dots, U_z^d)\}_{z \in \mathbb{R}_+^l}$ on a given probability space, where $d, l \in \mathbb{N}$, is *asymptotically independent* if the joint CDF is asymptotically equal to a product of the CDFs of the components: i.e. for any $a_i \in (-\infty, \infty]$, $i = 1, \dots, d$, it holds

$$P(U_z^1 \leq a_1, \dots, U_z^d \leq a_d) = \prod_{i=1}^d P(U_z^i \leq a_i) + o(1), \text{ as } \min\{z_i\} \rightarrow \infty$$

Asymptotic distributions

Main result

- ▶ **Definition.** A family of random vectors $\{(U_z^1, \dots, U_z^d)\}_{z \in \mathbb{R}_+^l}$ on a given probability space, where $d, l \in \mathbb{N}$, is *asymptotically independent* if the joint CDF is asymptotically equal to a product of the CDFs of the components: i.e. for any $a_i \in (-\infty, \infty]$, $i = 1, \dots, d$, it holds

$$P(U_z^1 \leq a_1, \dots, U_z^d \leq a_d) = \prod_{i=1}^d P(U_z^i \leq a_i) + o(1), \text{ as } \min\{z_i\} \rightarrow \infty$$

- ▶ **Notation:** ϕ : the Laplace exponent of the increasing ladder-height process H of X

Asymptotic distributions

Main result

- ▶ **Definition.** A family of random vectors $\{(U_z^1, \dots, U_z^d)\}_{z \in \mathbb{R}_+^l}$ on a given probability space, where $d, l \in \mathbb{N}$, is *asymptotically independent* if the joint CDF is asymptotically equal to a product of the CDFs of the components: i.e. for any $a_i \in (-\infty, \infty]$, $i = 1, \dots, d$, it holds

$$P(U_z^1 \leq a_1, \dots, U_z^d \leq a_d) = \prod_{i=1}^d P(U_z^i \leq a_i) + o(1), \text{ as } \min\{z_i\} \rightarrow \infty$$

- ▶ **Notation:** ϕ : the Laplace exponent of the increasing ladder-height process H of X

- ▶ **Theorem.**

The triplet $\{(Y(t), Z(x+y), M(t,x))\}_{t,x,y \in \mathbb{R}_+}$ is asymptotically independent and the weak limit $Z(x) \xrightarrow{\mathcal{D}} Z(\infty)$, as $x \rightarrow \infty$, exists, where

$$E[e^{-vZ(\infty)}] = \frac{\gamma}{\gamma + v} \cdot \frac{\phi(v)}{\phi(0)} \quad \text{for all } v \in \mathbb{R}_+.$$

Asymptotic distributions

Asymptotic distributions

► **Theorem.**

The triplet $\{(Y(t), Z(x+y), M(t, x))\}_{t, x, y \in \mathbb{R}_+}$ is asymptotically independent and the weak limit $Z(x) \xrightarrow{\mathcal{D}} Z(\infty)$ exists, as $x \rightarrow \infty$, where

$$E[e^{-vZ(\infty)}] = \frac{\gamma}{\gamma + v} \cdot \frac{\phi(v)}{\phi(0)} \quad \text{for all } v \in \mathbb{R}_+.$$

Asymptotic distributions

► **Theorem.**

The triplet $\{(Y(t), Z(x+y), M(t,x))\}_{t,x,y \in \mathbb{R}_+}$ is asymptotically independent and the weak limit $Z(x) \xrightarrow{\mathcal{D}} Z(\infty)$ exists, as $x \rightarrow \infty$, where

$$E[e^{-vZ(\infty)}] = \frac{\gamma}{\gamma + v} \cdot \frac{\phi(v)}{\phi(0)} \quad \text{for all } v \in \mathbb{R}_+.$$

Remarks.

- In general, the functionals $M(t,x)$ and $Z(x)$ are not asymptotically independent as $t \wedge x \rightarrow \infty$. However, the Theorem implies that for any $\alpha > 1$ the variables $M(t,x)$ and $Z(\alpha x)$ are asymptotically independent as $t \wedge x \rightarrow \infty$.
- If the Lévy measure of X is heavy-tailed, the "single-large jump"-heuristic suggests that $Z(x+y)$ and $M(t,x)$ are not asymptotically independent.

Asymptotic distributions

► **Theorem.**

The triplet $\{(Y(t), Z(x+y), M(t, x))\}_{t, x, y \in \mathbb{R}_+}$ is asymptotically independent and the weak limit

$Z(x) \xrightarrow{\mathcal{D}} Z(\infty)$, as $x \rightarrow \infty$ exists, where

$$E[e^{-vZ(\infty)}] = \frac{\gamma}{\gamma + v} \cdot \frac{\phi(v)}{\phi(0)} \quad \text{for all } v \in \mathbb{R}_+.$$

What about the marginal asymptotic distributions of $Y(t)$ and $M(t, x)$?

Asymptotic distributions

► **Theorem.**

The triplet $\{(Y(t), Z(x+y), M(t,x))\}_{t,x,y \in \mathbb{R}_+}$ is asymptotically independent and the weak limit

$Z(x) \xrightarrow{\mathcal{D}} Z(\infty)$, as $x \rightarrow \infty$ exists, where

$$E[e^{-vZ(\infty)}] = \frac{\gamma}{\gamma + v} \cdot \frac{\phi(v)}{\phi(0)} \quad \text{for all } v \in \mathbb{R}_+.$$

What about the marginal asymptotic distributions of $Y(t)$ and $M(t,x)$?

- The Cramér's condition implies that X tends to $-\infty$ almost surely, and hence, by the classical time reversal argument, the reflected process Y has a stationary distribution $Y(\infty)$ equal to the law of the ultimate supremum $\sup_{t \geq 0} X(t)$.

Asymptotic distributions

► **Theorem.**

The triplet $\{(Y(t), Z(x+y), M(t,x))\}_{t,x,y \in \mathbb{R}_+}$ is asymptotically independent and the weak limit $Z(x) \xrightarrow{\mathcal{D}} Z(\infty)$, as $x \rightarrow \infty$ exists, where

$$E[e^{-vZ(\infty)}] = \frac{\gamma}{\gamma + v} \cdot \frac{\phi(v)}{\phi(0)} \quad \text{for all } v \in \mathbb{R}_+.$$

What about the marginal asymptotic distributions of $Y(t)$ and $M(t,x)$?

- The Cramér's condition implies that X tends to $-\infty$ almost surely, and hence, by the classical time reversal argument, the reflected process Y has a stationary distribution $Y(\infty)$ equal to the law of the ultimate supremum $\sup_{t \geq 0} X(t)$.
- Doney & Maller (2005)'s Cramér estimate under the excursion measure implies that $M(t,x)$ converges weakly to a Gumbel distribution if the quantity $te^{-\gamma x}$ tends to a positive constant as $t, x \uparrow \infty$

Asymptotic distributions

- ▶ **Corollary 1.** Let $m \doteq \lim_{u \rightarrow \infty} \phi(u)/u$, H upcrossing ladder height process, and $\bar{\nu}_H(x) \doteq \nu_H((x, \infty))$, $x > 0$. Then the law of the asymptotic overshoot $Z(\infty)$ is given by:

$$P(Z(\infty) > x) = \frac{\gamma}{\phi(0)} e^{-\gamma x} \int_x^\infty e^{\gamma y} \bar{\nu}_H(y) dy, \quad x \in [0, \infty).$$

In particular, $Z(\infty)$ is a continuous random variable except possibly at the origin.

Asymptotic distributions

- ▶ **Corollary 1.** Let $m \doteq \lim_{u \rightarrow \infty} \phi(u)/u$, H upcrossing ladder height process, and $\bar{\nu}_H(x) \doteq \nu_H((x, \infty))$, $x > 0$. Then the law of the asymptotic overshoot $Z(\infty)$ is given by:

$$P(Z(\infty) > x) = \frac{\gamma}{\phi(0)} e^{-\gamma x} \int_x^\infty e^{\gamma y} \bar{\nu}_H(y) dy, \quad x \in [0, \infty).$$

In particular, $Z(\infty)$ is a continuous random variable except possibly at the origin.

- ▶ **Corollary 2.** The weak limit of the random vector $(Y(t), Z(x))$, as $x \wedge t \rightarrow \infty$, exists and the law $(Y(\infty), Z(\infty))$ is determined by the joint Laplace transform

$$E[\exp(-uY(\infty) - vZ(\infty))] = \frac{\gamma}{\gamma + v} \cdot \frac{\phi(v)}{\phi(u)} \quad \text{for all } u, v \in \mathbb{R}_+.$$

Asymptotic distributions

- ▶ **Corollary 1.** Let $m \doteq \lim_{u \rightarrow \infty} \phi(u)/u$, H upcrossing ladder height process, and $\bar{\nu}_H(x) \doteq \nu_H((x, \infty))$, $x > 0$. Then the law of the asymptotic overshoot $Z(\infty)$ is given by:

$$P(Z(\infty) > x) = \frac{\gamma}{\phi(0)} e^{-\gamma x} \int_x^\infty e^{\gamma y} \bar{\nu}_H(y) dy, \quad x \in [0, \infty).$$

In particular, $Z(\infty)$ is a continuous random variable except possibly at the origin.

- ▶ **Corollary 2.** The weak limit of the random vector $(Y(t), Z(x))$, as $x \wedge t \rightarrow \infty$, exists and the law $(Y(\infty), Z(\infty))$ is determined by the joint Laplace transform

$$E[\exp(-uY(\infty) - vZ(\infty))] = \frac{\gamma}{\gamma + v} \cdot \frac{\phi(v)}{\phi(u)} \quad \text{for all } u, v \in \mathbb{R}_+.$$

Asymptotic distributions

Notation.

- ▶ $\hat{\phi}$: Laplace exponent of the decreasing ladder-height process
- ▶ \hat{L}^{-1} : decreasing ladder-time process
- ▶ $\Gamma(\cdot)$: gamma function

Asymptotic distributions

Notation.

- ▶ $\hat{\phi}$: Laplace exponent of the decreasing ladder-height process
- ▶ \hat{L}^{-1} : decreasing ladder-time process
- ▶ $\Gamma(\cdot)$: gamma function
- ▶ **Corollary 3.** Assume that $\lim_{t \uparrow \infty} t e^{-\gamma x} = \lambda$ for some $\lambda > 0$. Then $(Y(t), Z(x+y), M(t, x))$ converges weakly as $t \wedge y \rightarrow \infty$ and the joint limit law $(Y(\infty), Z(\infty), M(\infty))$ is given by the Fourier-Laplace transform:

$$E[\exp(-uY(\infty) - vZ(\infty) + i\beta M(\infty))] = \frac{\gamma}{\gamma + v} \cdot \frac{\phi(v)}{\phi(u)} \\ \cdot \Gamma\left(1 - \frac{i\beta}{\gamma}\right) \cdot \exp\left[i\beta\gamma^{-1} \log\left(\lambda \ell C_\gamma \hat{\phi}(\gamma)\right)\right]$$

for all $u, v \in \mathbb{R}_+$, $\beta \in \mathbb{R}$, where $\ell \doteq 1/E[\hat{L}^{-1}(1)]$ and $C_\gamma \doteq \frac{\phi(0)}{\gamma\phi'(-\gamma)}$.

Sketch of the proof: asymptotic independence

Sketch of the proof: asymptotic independence

- ▶ **Key observation:** events translated in terms of **Ito-excursion process** ϵ of Y which forms a Poisson point process indexed by local time \hat{L} of Y at zero.

Sketch of the proof: asymptotic independence

- ▶ **Key observation:** events translated in terms of **Ito-excursion process** ϵ of Y which forms a Poisson point process indexed by local time \widehat{L} of Y at zero.
- ▶ **Splitting property of a Poisson point process** η at first entrance into A :

$$\eta^A := \{\eta_t : t < H_A\} \perp \eta(H_A)$$

with

$$H_A = \inf\{t \geq 0 : \eta(t) \in A\}.$$

Sketch of the proof: asymptotic independence

- ▶ **Key observation:** events translated in terms of **Ito-excursion process** ϵ of Y which forms a Poisson point process indexed by local time \widehat{L} of Y at zero.
- ▶ **Splitting property of a Poisson point process** η at first entrance into A :

$$\eta^A := \{\eta_t : t < H_A\} \perp \eta(H_A)$$

with

$$H_A = \inf\{t \geq 0 : \eta(t) \in A\}.$$

- ▶ In fact, as in Greenwood & Pitman (1980) we will also consider the excursion process $\epsilon' = (\epsilon, \tilde{\epsilon})$ of (Y, N^q) away from $\{0\} \times \mathbb{R}_+$ where N^q is an independent Poisson process with rate q .

Sketch of the proof: asymptotic independence

Sketch of the proof: asymptotic independence

- ▶ For example, splitting at $H_{B'}$ with $B' = \{\epsilon' : \rho(x, \epsilon) < \zeta(\epsilon)\}$ and noting $\widehat{L}(\tau(x)) = H_{B'}$ yields

$$\begin{aligned} P(\widehat{L}(\tau(x)) > \widehat{L}(e_q), Y(e_q) \in A, Z(x) \in B) \\ = P(\widehat{L}(\tau(x)) > \widehat{L}(e_q), Y(e_q) \in A)P(Z(x) \in B), \end{aligned}$$

where e_q is an independent exponential time with mean q^{-1} .

Sketch of the proof: asymptotic independence

- ▶ For example, splitting at $H_{B'}$ with $B' = \{\epsilon' : \rho(x, \epsilon) < \zeta(\epsilon)\}$ and noting $\widehat{L}(\tau(x)) = H_{B'}$ yields

$$\begin{aligned} P(\widehat{L}(\tau(x)) > \widehat{L}(e_q), Y(e_q) \in A, Z(x) \in B) \\ = P(\widehat{L}(\tau(x)) > \widehat{L}(e_q), Y(e_q) \in A)P(Z(x) \in B), \end{aligned}$$

where e_q is an independent exponential time with mean q^{-1} .

- ▶ Splitting at $H_{A'}$ with $A' = \{\epsilon' = (\epsilon, \tilde{\epsilon}) : H_{\{1\}}(\tilde{\epsilon}) < \zeta(\epsilon)\}$ yields

$$\begin{aligned} P(\widehat{L}(\tau(x)) > \widehat{L}(e_q), Y(e_q) \in A) \\ = P(\widehat{L}(\tau(x)) > \widehat{L}(e_q))P(Y(e_q) \in A). \end{aligned}$$

Sketch of the proof: asymptotic independence

- ▶ For example, splitting at $H_{B'}$ with $B' = \{\epsilon' : \rho(x, \epsilon) < \zeta(\epsilon)\}$ and noting $\widehat{L}(\tau(x)) = H_{B'}$ yields

$$\begin{aligned} P(\widehat{L}(\tau(x)) > \widehat{L}(e_q), Y(e_q) \in A, Z(x) \in B) \\ = P(\widehat{L}(\tau(x)) > \widehat{L}(e_q), Y(e_q) \in A)P(Z(x) \in B), \end{aligned}$$

where e_q is an independent exponential time with mean q^{-1} .

- ▶ Splitting at $H_{A'}$ with $A' = \{\epsilon' = (\epsilon, \tilde{\epsilon}) : H_{\{1\}}(\tilde{\epsilon}) < \zeta(\epsilon)\}$ yields

$$\begin{aligned} P(\widehat{L}(\tau(x)) > \widehat{L}(e_q), Y(e_q) \in A) \\ = P(\widehat{L}(\tau(x)) > \widehat{L}(e_q))P(Y(e_q) \in A). \end{aligned}$$

- ▶ The proof is based on such splitting arguments, inversion of the Laplace transforms and estimates on such as the following:

Lemma. The following limits hold, as $x \wedge t \rightarrow \infty$:

$$P(\widehat{L}(t) = \widehat{L}(\tau(x))) \rightarrow 0$$

$$\limsup_{x \wedge t \rightarrow \infty} P(\widehat{L}(t(1 - \delta_1)) \leq \widehat{L}(\tau(x)) \leq \widehat{L}(t(1 - \delta_2))) \leq \frac{8(\delta_1 \vee \delta_2)}{e}$$

Sketch of the proof: asymptotic distribution of overshoot

Sketch of the proof: asymptotic distribution of overshoot

- ▶ **Key observation:** overshoot expressed in terms of **Ito-excursion process** ϵ of Y which forms a Poisson point process indexed by local time \hat{L} of Y at zero.

Sketch of the proof: asymptotic distribution of overshoot

- ▶ **Key observation:** overshoot expressed in terms of **Ito-excursion process** ϵ of Y which forms a Poisson point process indexed by local time \widehat{L} of Y at zero.
- ▶ Poisson point process ϵ with intensity measure n satisfies

$$P(\epsilon(H_A) \in B) = n(B|A) := n(B)/n(A), \quad B \subset A, n(A) > 0,$$

where $H_A = \inf\{t \geq 0 : \epsilon(t) \in A\}$.

Sketch of the proof: asymptotic distribution of overshoot

- ▶ **Key observation:** overshoot expressed in terms of **Ito-excursion process** ϵ of Y which forms a Poisson point process indexed by local time \widehat{L} of Y at zero.
- ▶ Poisson point process ϵ with intensity measure n satisfies

$$P(\epsilon(H_A) \in B) = n(B|A) := n(B)/n(A), \quad B \subset A, n(A) > 0,$$

where $H_A = \inf\{t \geq 0 : \epsilon(t) \in A\}$.

- ▶ Taking

$$A = \{\epsilon : \rho(x, \epsilon) < \zeta(\epsilon)\}, \quad B = \{\epsilon : \epsilon(\rho(x, \epsilon)) - x > y, \rho(x, \epsilon) < \zeta(\epsilon)\}$$

with $\rho(x, \epsilon) = \inf\{t \geq 0 : \epsilon(t) > x\}$, we have

$$P(Z(x) > y) = n(\epsilon(\rho(x, \epsilon)) - x > y | \rho(x, \epsilon) < \zeta(\epsilon)).$$

Sketch of the proof: asymptotic distribution of overshoot

- ▶ **Key observation:** overshoot expressed in terms of **Ito-excursion process** ϵ of Y which forms a Poisson point process indexed by local time \widehat{L} of Y at zero.

- ▶ Poisson point process ϵ with intensity measure n satisfies

$$P(\epsilon(H_A) \in B) = n(B|A) := n(B)/n(A), \quad B \subset A, n(A) > 0,$$

where $H_A = \inf\{t \geq 0 : \epsilon(t) \in A\}$.

- ▶ Taking

$$A = \{\epsilon : \rho(x, \epsilon) < \zeta(\epsilon)\}, \quad B = \{\epsilon : \epsilon(\rho(x, \epsilon)) - x > y, \rho(x, \epsilon) < \zeta(\epsilon)\}$$

with $\rho(x, \epsilon) = \inf\{t \geq 0 : \epsilon(t) > x\}$, we have

$$P(Z(x) > y) = n(\epsilon(\rho(x, \epsilon)) - x > y | \rho(x, \epsilon) < \zeta(\epsilon)).$$

- ▶ **Lemma.** As $x \rightarrow \infty$, we have

$$n(e^{-v(\epsilon(\rho(x)) - x)} | \rho(x) < \zeta) \rightarrow \frac{\gamma}{\gamma + v} \cdot \frac{\phi(v)}{\phi(0)}.$$