The Law of the Iterated Logarithm and Small Deviations of stable Lévy processes

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Two metrics of the Skorokhod space

Let $D[0,1]$ be the space of càdlàg functions.
Metric generated by the uniform distance:

$$d_{un}(f,g) = \sup_{t\in[0,1]} |f(t) - g(t)| = \|f - g\|$$

Metric generated by the Skorokhod distance:

$$d_{sk}(f,g) = \inf_{h\in H} \max\{\|h - Id\|, \|f(\cdot) - g(h(\cdot))\|\},$$

where $H = \{h : [0,1] \to [0,1], \mathit{increasing}\}$

$$d_{sk} \leq d_{un}$$

Shifted Small Deviations

Consider a Lévy process $\{X(t), t \in [0,1]\}$.

Small deviations deal with

$$P\{X(\cdot) \in B(f,r)\} =? \quad as\ r \to 0,$$

$B(f,r)$ is a ball of radius r with the center at f.

$f \in D[0,1] \setminus C[0,1]$ The uniform small balls are empty.
Open problem: the measure of the Skorokhod topology balls.

$f \in C[0,1]$ Uniform topology - settings of this talk.

Symmetric α-stable Lévy processes

Let X_α be a symmetric α-stable Lévy process (SαS), $\alpha \in (0,2)$

$$E \exp\{i\, z\, X_\alpha(t)\} = \exp\{-t|z|^\alpha\}.$$

Its Lévy measure is symmetric and is equal to
$d\Lambda(\ell) = |\ell|^{-(1+\alpha)} d\ell$.

Small deviations:

$$P\{\|X_\alpha(\cdot) - f(\cdot)\| < r\} = ?$$

$r \to 0$, $f \in Supp\, \mathrm{P}_{X_\alpha}$

Supports of processes in functional spaces

Definition

Let X be a càdlàg process, P_X be the distribution of X in $D[0,1]$. We say that $f \in Supp\,\mathrm{P}_X$ iff $P\{\|X - f\| < r\} > 0$ for any $r > 0$.

For example, for the Wiener process W it is well known fact that its support is equal to $\{f \in C[0,1] : f(0) = 0\}$.

Ishikawa (2002): Support of any SαS Lévy process is also $\{f \in C[0,1] : f(0) = 0\}$. , i.e., for any such f we have $P\{\|X_\alpha - f\| < r\} > 0$ for any $r > 0$.

Centered Small Deviations for SαS process

For $f \equiv 0$, the well-known result by A.Mogulskii (1974) is:

$$P\left\{\|X_\alpha\| < r\right\} = \exp\left\{-K_\alpha \frac{1}{r^\alpha}(1 + o(1))\right\} \text{ as } \quad r \to 0.$$

Here $0 < K_\alpha < \infty$, for which there is an implicit expression.

Onsager-Machlup functional

The most probable sample path problem:

$$J(f) = \lim_{r \to 0} \frac{P\{\|X - f\| < r\}}{P\{\|X\| < r\}}.$$

For diffusion processes this problem is quite well studied and $J(f)$ is described via so called Onsager-Machlup functional.

As far as I know this problem is open for Lévy processes.

Shifted small deviations for SαS

Theorem (Shmileva '09)

Let $\alpha \in (1,2)$*. For any* f *such that* $f \in C[0,1]$ *and* $f(0) = 0$ *we have*

$$P\{\|X_\alpha(\cdot) - \lambda f(\cdot)\| < r\} = \exp\left\{-K_\alpha \frac{1}{r^\alpha}(1 + o(1))\right\},$$

as $r \to 0$, $\lambda r^{\alpha-1} \to 0$*. The constant* K_α *is from the Mogulskii estimate.*

Proof scheme

Upper bound uses the Anderson inequality

$$P\{\|X_\alpha(\cdot) - \lambda f(\cdot)\| < r\} \leq P\{\|X_\alpha\| < r\}.$$

Lower bound:

- Girsanov transform for additive processes.
- Martingal condition.
- Centered small deviations for martingals from the domain of attraction of SαS Lévy process are known from Moguskii paper.

"Large" shifts

What will happen if $\lambda r^{\alpha-1} \to \infty$?

The result is known just for $f(\cdot) = Id(\cdot)$:

$$P\left\{\|X_\alpha(\cdot) - \lambda Id(\cdot)\| < r\right\} = \exp\left\{-C_\alpha \frac{\lambda}{r} \log \lambda r^{\alpha-1}(1 + o(1))\right\}.$$

This could be obtained by the method from Aurzada, Dereich (2009). Their method allows to find small deviation order $F(r)$ for centered balls for general Lévy process

$$P\{\|X\| < r\} = \exp\{-C \cdot F(r)(1 + o(1))\}.$$

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Functional Law of the Iterated Logarithm (LIL). General settings.

Let $X(t), t \in (0, \infty)$ be a Lévy process. Put

$$Y_T(t) := \frac{X(T\,t)}{\varphi(T)},\ \ t \in [0,1].$$

Functional LIL studies the limiting behavior of the family of scalings $\{Y_T(\cdot)\}_{T>0}$ for $T \to \infty$:

$$\{Y_T(t)\ t \in [0,1]\}_{T>0} \to\to \mathcal{S}\ \ a.s.$$

Namely, the a.s. cluster set $\mathcal{S} \subseteq C[0,1]$ is studied depending on the scaling function φ.

Formalization of the FLIL problem

Functional LIL: $\left\{ \frac{X(Tt)}{\varphi(T)},\ t \in [0,1] \right\}_{T>0} \to\to \mathcal{S}\ \ a.s.$

is equivalent to the following two statements:

1. $\lim_{T\to\infty} \inf_{f\in\mathcal{S}} \| \frac{X(T\,\cdot)}{\varphi(T)} - f(\cdot)\| = 0$ a.s.
2. for all $f \in \mathcal{S}$ $\liminf_{T\to\infty} \| \frac{X(T\,\cdot)}{\varphi(T)} - f(\cdot)\| = 0$ a.s.

Functional LIL for the Wiener process

Strassen(1964):

$$\left\{ \frac{W(T\,t)}{\sqrt{2T\ log\ log\ T}},\ t \in [0,1] \right\}_{T>0} \to\to \mathrm{S} \ \ a.s.$$

$$\mathrm{S} := \left\{ f : f(0) = 0,\ f \in \mathrm{AC}[0,1],\ \int_0^1 f'(t)^2\,dt \le 1 \right\}.$$

Consequence: Put $t = 1$ to obtain the classical LIL

$$\limsup_{T\to\infty} \frac{W(T)}{\sqrt{2T\ log\ log\ T}} = 1 \ \ a.s.$$

Functional LIL statements for SαS Lévy process

Theorem (Shmileva'09)

Let X_α be SαS Lévy process and $\alpha \in (1,2)$. For any $f \in C[0,1]$, $f(0) = 0$ it holds

$$\liminf_{T\to\infty} \left\| \frac{X_\alpha(T\cdot)}{T^{1/\alpha} c(\log\log T)^{-1/\alpha}} - f(\cdot) \right\| \geq \frac{K_\alpha^{1/\alpha}}{c} \quad a.s.$$

Corollary: No a.s. cluster set for these scalings. The scaling is too weak.

Functional LIL statements for SαS Lévy process

Theorem (Shmileva'09)

For $\delta \in (0,1]$ it holds any $f \in C[0,1]$, $f(0) = 0$

$$\liminf_{T\to\infty} (\log\log T)^{\delta} \left\| \frac{X_\alpha(T\cdot)}{T^{1/\alpha}(\log\log T)^{\delta-1/\alpha}} - f(\cdot) \right\| = K_\alpha^{1/\alpha} \quad a.s.$$

Corollaries

- The a.s. cluster set of

$$\left\{ \frac{X_\alpha(T\,t)}{T^{1/\alpha}(\log\log T)^{\delta-1/\alpha}}, t \in [0,1] \right\}_{T>0} \to\to C \quad a.s.$$

  is equal to the set of all admissible functions
  $C := \{f \in C[0,1] : f(0) = 0\}$.
- The rate of convergence to the admissible functions.
- Put $f \equiv 0$ to obtain the Chung LIL:

$$\liminf_{T\to\infty} \frac{\|X_\alpha(T\cdot)\|}{(T/\log\log T)^{1/\alpha}} = K_\alpha^{1/\alpha} \;\; a.s.$$

No classical LIL statement for α-stable process

Integral test by Bertoin(1996):

$$\limsup_{T\to\infty} \frac{|X_\alpha(T)|}{T^{1/\alpha}h(T)} = 0 \quad \text{or } \infty \text{ a.s.}$$

according as

$$\int_1^\infty \frac{dt}{t\,h(t)^\alpha} < \infty \quad \text{or } = \infty.$$

Examples of the INFINITELY often crossed levels:
$t^{1/\alpha}(\log t)^{1/\alpha}$, $t^{1/\alpha}(\log\log t)^a$, $t^{1/\alpha}$.

Examples of the FINITELY often crossed levels:
$t^{1/\alpha}(\log t)^{1/\alpha+\epsilon}$, $t^{1/\alpha+\epsilon}$.

Open questions

- Shifted Small Deviations for "large" shifts, i.e. under $\lambda r^{\alpha-1} \to \infty$ for arbitrary f. This is equivalent to the small deviations for additive processes.
- Functional LIL for $\left\{ \frac{X_\alpha(T\,t)}{T^{1/\alpha}h(T)}, t \in [0,1] \right\}_{T>0}$, $h(T) >> (\log\log T)^{1-1/\alpha}$
- Are there non-trivial cluster sets?
- Shifted small deviations for different topologies.
- The most probable path calculations (Onsager-Machlup functional for Lévy processes).

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Thank you for your attention!