Stationary Solutions of Spatial ARMA Equations

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Overview

Spatial ARMA model

Let $d \in \mathbb{N}$ and consider the difference equation

$$Y_{\mathbf{t}} - \sum_{\mathbf{n} \in R} \phi_{\mathbf{n}} Y_{\mathbf{t}-\mathbf{n}} = Z_{\mathbf{t}} + \sum_{\mathbf{n} \in S} \theta_{\mathbf{n}} Z_{\mathbf{t}-\mathbf{n}}, \quad \mathbf{t} \in \mathbb{Z}^d. \tag{1}$$

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Define the polynomials

$$\begin{aligned} \Phi(\mathbf{z}) &= 1 - \sum_{\mathbf{n} \in R} \phi_{\mathbf{n}} \mathbf{z}^{\mathbf{n}}, \quad \text{and} \\ \Theta(\mathbf{z}) &= 1 + \sum_{\mathbf{k} \in S} \theta_{\mathbf{k}} \mathbf{z}^{\mathbf{k}}, \quad \mathbf{z} = (z_1, \ldots, z_d) \in \mathbb{C}^d. \end{aligned}$$

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We are interested in strictly stationary solutions $(Y_{\mathbf{t}})_{\mathbf{t} \in \mathbb{Z}^d}$ of equation (1).

Time series ARMA model: $d = 1$

Theorem (Brockwell and Lindner (2010))

Suppose $\Phi(z)$ and $\Theta(z)$ have no common zeros. Then the ARMA model admits a strictly stationary solution $Y = (Y_t)_{t\in\mathbb{Z}}$ if and only if

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If (i) or (ii) holds, the unique strictly stationary solution is given by

$$Y_t = \sum_{k\in\mathbb{Z}} \psi_k Z_{t-k}, \quad t \in \mathbb{Z},$$

where $\Theta(z)/\Phi(z) = \sum_{k\in\mathbb{Z}} \psi_k z^k$ is the Laurent expansion in some annulus containing the unit circle.

Weakly stationary solutions, $d \geq 2$

Theorem

The spatial ARMA model admits a weakly stationary solution if and only if

$$\frac{\Theta(e^{-i\cdot})}{\Phi(e^{-i\cdot})} \in L^2(\mathbb{T}^d),$$

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$$\int_{\mathbb{T}^d} \left| \frac{\Theta(e^{-i\mathbf{t}})}{\Phi(e^{-i\mathbf{t}})} \right|^2 d\lambda^d(\mathbf{t}) < \infty, \quad \mathbb{T} = (-\pi, \pi].$$

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If this condition is fulfilled, then a weakly stationary solution is given by

$$Y_{\mathbf{t}} = \sum_{\mathbf{k} \in \mathbb{Z}^d} \psi_{\mathbf{k}} Z_{\mathbf{t}-\mathbf{k}}, \quad \mathbf{t} \in \mathbb{Z}^d,$$

where $\Theta(e^{-i\mathbf{t}})/\Phi(e^{-i\mathbf{t}}) = \sum_{\mathbf{k} \in \mathbb{Z}^d} \psi_{\mathbf{k}} e^{-i\mathbf{k}\mathbf{t}}$ is the *Fourier expansion* of $\Theta(e^{-i\cdot})/\Phi(e^{-i\cdot})$.

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$$U_{\mathbf{t}} = \Phi(\mathbf{B})Y_{\mathbf{t}}, \qquad V_{\mathbf{t}} = \Theta(\mathbf{B})Z_{\mathbf{t}}, \qquad \mathbf{t} \in \mathbb{Z}^d.$$

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Then U and V are weakly stationary and their spectral measures fulfill by the ARMA equation

$$\left|\Phi(e^{-i\mathbf{t}})\right|^2 d\mu_Y(\mathbf{t}) = \left|\Theta(e^{-i\mathbf{t}})\right|^2 \frac{\sigma^2}{(2\pi)^d} d\lambda^d(\mathbf{t}), \qquad \mathbf{t} \in \mathbb{T}^d.$$

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Define $N := \{\mathbf{t}\in\mathbb{T}^d : \Phi(e^{-i\mathbf{t}}) = 0\}$

$$\mu_Y\left(\mathbb{T}^d\backslash N\right) = \frac{\sigma^2}{(2\pi)^d}\int_{\mathbb{T}^d\backslash N}\left|\frac{\Theta(e^{-i\mathbf{t}})}{\Phi(e^{-i\mathbf{t}})}\right|^2 \, d\lambda^d(\mathbf{t}) < \infty.$$

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Notice that $\lambda^d(N) = 0$.

Example

Consider the polynomials

$$\begin{aligned}\Theta(z_1, z_2) &= (1 - z_1)(1 - z_2), \\ \Phi(z_1, z_2) &= 1 - \frac{1}{2}z_1 - \frac{1}{2}z_2, \quad z_1, z_2 \in \mathbb{C}.\end{aligned}$$

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It is easy to show that

$$\frac{\Theta(e^{-i\cdot})}{\Phi(e^{-i\cdot})}\in L^2(\mathbb{T}^2),$$

even though $\Phi(1,1)=0$.

Linear strictly stationary solutions

Definition

A random field $(Y_{\mathbf{t}})_{\mathbf{t}\in\mathbb{Z}^d}$, which solves the ARMA equation, is called *linear strictly stationary solution*, if there are coefficients $(\psi_{\mathbf{k}})_{\mathbf{k}\in\mathbb{Z}^d} \subset \mathbb{C}$, such that

$$Y_{\mathbf{t}} = \sum_{\mathbf{k}\in\mathbb{Z}^d} \psi_{\mathbf{k}} Z_{\mathbf{t}-\mathbf{k}}, \quad \mathbf{t} \in \mathbb{Z}^d,$$

where the right-hand side converges almost surely absolutely.

Necessary condition

Theorem

If the ARMA equation admits a linear strictly stationary solution, then

$$\frac{\Theta(e^{-i\cdot})}{\Phi(e^{-i\cdot})} \in L^2(\mathbb{T}^d).$$

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Thus, by the Riesz-Fischer Theorem

$$f(e^{-i\mathbf{t}}) = \sum_{\mathbf{k}\in\mathbb{Z}^d} \psi_{\mathbf{k}} e^{-i\mathbf{k}\mathbf{t}}, \quad \mathbf{t} \in \mathbb{T}^d,$$

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defines a function in $L^2(\mathbb{T}^d)$.

Show that $\frac{\Theta(e^{-i\cdot})}{\Phi(e^{-i\cdot})} = f(e^{-i\cdot})$.

Sufficient condition

Theorem

If the AR polynomial $\Phi(e^{-i\cdot})$ has no zero on $\mathbb{T}^d$, then in some polyannulus a Laurent expansion of $\Theta(\boldsymbol{z})/\Phi(\boldsymbol{z})$ exists given by

$$\frac{\Theta(\boldsymbol{z})}{\Phi(\boldsymbol{z})} = \sum_{\boldsymbol{k}\in\mathbb{Z}^d} \psi_{\boldsymbol{k}} \boldsymbol{z}^{\boldsymbol{k}}, \quad \boldsymbol{z} \in \{(y_1, \dots, y_d) \in \mathbb{C}^d : r_i < |y_i| < R_i\},$$
$$r_i \in (0,1), \quad R_i > 1.$$

If further $\boldsymbol{E}\log_+^d |Z_1| < \infty$, then

$$Y_t := \sum_{\boldsymbol{k}\in\mathbb{Z}^d} \psi_{\boldsymbol{k}} Z_{\boldsymbol{t}-\boldsymbol{k}}, \quad \boldsymbol{t} \in \mathbb{Z}^d,$$

converges almost surely absolutely.

Causal solutions

Define the index set

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A strictly stationary random field $(Y_{\mathbf{t}})_{\mathbf{t} \in \mathbb{Z}^d}$, which fulfills the ARMA equation, is called *causal solution* of the spatial ARMA model, if $Y_{\mathbf{t}}$ is measurable with respect to $\sigma(Z_{\mathbf{s}} : \mathbf{s} \leq \mathbf{t})$ for each $\mathbf{t} \in \mathbb{Z}^d$.

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The Hilbert space of all functions $f : \mathbb{C}^d \to \mathbb{C}$, which are holomorphic in $\mathbb{D}^d = \{\mathbf{z} \in \mathbb{C}^d : |z_i| < 1, \ i = 1, \dots, d\}$

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$$\lim_{r \to 1} \int_{\mathbb{T}^d} \left| f(re^{i\mathbf{t}}) \right|^2 d\lambda^d(\mathbf{t}) < \infty,$$

is denoted by $H^2 \subset L^2(\mathbb{T}^d)$ and named *Hardy space*.

Theorem

Assume that $(Z_{\boldsymbol{t}})_{\boldsymbol{t}\in\mathbb{Z}^d}$ is nondeterministic. If the ARMA equation admits a causal solution $(Y_{\boldsymbol{t}})_{\boldsymbol{t}\in\mathbb{Z}^d}$, then

$$\frac{\Theta(z)}{\Phi(z)} \in H^2.$$

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denote the power series expansion of $\Theta(\boldsymbol{z})/\Phi(\boldsymbol{z})$, then the solution $(Y_{\boldsymbol{t}})_{\boldsymbol{t}\in\mathbb{Z}^d}$ is given by $Y_{\boldsymbol{t}} = \sum_{\boldsymbol{k}\in\mathbb{N}_0^d} \alpha_{\boldsymbol{k}} Z_{\boldsymbol{t}-\boldsymbol{k}}$, where the convergence of the right hand side is almost surely rectangular.

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Definition

Let $(\psi_{\mathbf{k}})_{\mathbf{k}\in\mathbb{N}_0^d} \subset \mathbb{C}$ and $(Z_{\mathbf{k}})_{\mathbf{k}\in\mathbb{N}_0^d}$ be a random field. The multiple series $\sum_{\mathbf{k}\in\mathbb{N}_0^d} \psi_{\mathbf{k}} Z_{\mathbf{k}}$ converges *almost surely in the rectangular sense*, if the limits

$$\lim_{n\to\infty} \sum_{k_1=0}^{N_{1n}} \cdots \sum_{k_d=0}^{N_{dn}} \psi_{\mathbf{k}} Z_{\mathbf{k}},$$

for all sequences $(N_{1n}, \ldots, N_{dn})_{n\in\mathbb{N}} \subset \mathbb{N}_0^d$ with $\min(N_{1n}, \ldots, N_{dn}) \to \infty$ $(n \to \infty)$ almost surely exist and coincide.

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For this notion of convergence Klesov(1995) established a generalization of the three series theorem of Kolmogorov.

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Suppose $(Y_{\mathbf{t}})_{\mathbf{t}\in\mathbb{Z}^d}$ is a causal solution and the distributions of Y and Z are symmetric.

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$$I_{\mathbf{N}} = \{\mathbf{k} \in \mathbb{Z}^d : 0 \leq k_i \leq N_i, i = 1, \ldots, d\}.$$

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Reorganizing the ARMA equation and replacing some random variables $Y_{\mathbf{t}}$ by the ARMA equation yields

$$\begin{aligned} Y_{\mathbf{t}} &= \sum_{\mathbf{n}\in I_{\mathbf{N}}} \alpha_{\mathbf{n},\mathbf{N}} Z_{\mathbf{t}-\mathbf{n}} + \sum_{\mathbf{n}\in B^S_{\mathbf{N}}} \beta_{\mathbf{n},\mathbf{N}} Z_{\mathbf{t}-\mathbf{n}} + \sum_{\mathbf{n}\in B^R_{\mathbf{N}}} \gamma_{\mathbf{n},\mathbf{N}} Y_{\mathbf{t}-\mathbf{n}} \\ &=: A_{\mathbf{t},\mathbf{N}} + B_{\mathbf{t},\mathbf{N}} + C_{\mathbf{t},\mathbf{N}}, \quad \mathbf{N} \in \mathbb{N}_0^d. \end{aligned}$$

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$$\begin{aligned} Y_{\mathbf{t}} &= \sum_{\mathbf{n}\in I_{\mathbf{N}}} \alpha_{\mathbf{n},\mathbf{N}} Z_{\mathbf{t}-\mathbf{n}} + \sum_{\mathbf{n}\in B_{\mathbf{N}}^S} \beta_{\mathbf{n},\mathbf{N}} Z_{\mathbf{t}-\mathbf{n}} + \sum_{\mathbf{n}\in B_{\mathbf{N}}^R} \gamma_{\mathbf{n},\mathbf{N}} Y_{\mathbf{t}-\mathbf{n}} \\ &=: A_{\mathbf{t},\mathbf{N}} + B_{\mathbf{t},\mathbf{N}} + C_{\mathbf{t},\mathbf{N}}, \quad \mathbf{N} \in \mathbb{N}_0^d. \end{aligned}$$

Show that $A_{\mathbf{t},\mathbf{N}_k}$ is not converging in probability to infinity as $\min(N_{1k}, \dots, N_{dk}) \to \infty (k \to \infty)$

Proof of the theorem.

Suppose $(Y_{\mathbf{t}})_{\mathbf{t}\in\mathbb{Z}^d}$ is a causal solution and the distributions of Y and Z are symmetric. Define for $\mathbf{N} = (N_1, \ldots, N_d) \in \mathbb{N}_0^d$ the index set

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$\implies \quad A_{\mathbf{t},\mathbf{N}_k}$ is converging a.s.

Some general comments I

Example

For $d = 5$ the function $\Phi^{-1}(\mathbf{z})$, where

$$\Phi(\mathbf{z}) = 1 - \frac{1}{5}\sum_{i=1}^{5} z_i, \quad \mathbf{z} = (z_1, \ldots, z_5) \in \mathbb{C}^5,$$

is square integrable, and thus $\Phi^{-1}(e^{i\mathbf{t}}) \in H^2 \subset L^2(\mathbb{T}^5)$.

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is square integrable, and thus $\Phi^{-1}(e^{i\mathbf{t}}) \in H^2 \subset L^2(\mathbb{T}^5)$. Notice that $\Phi(1, \ldots, 1) = 0$.

Some general comments II

For $d = 2$ this is not possible:

By mean value theorem if $\Phi(e^{i\mathbf{w}}) = 0$ for some $\mathbf{w} = (w_1, w_2) \in \mathbb{T}^2$, then for arbitrary norm $||\cdot||$ on $\mathbb{R}^2$ and some $C > 0$

$$|\Phi(e^{i\mathbf{w}+\mathbf{h}})| = |\Phi(e^{i\mathbf{w}+\mathbf{h}}) - \Phi(e^{i\mathbf{w}})| \leq C||\mathbf{h}||, \quad \mathbf{h} = (h_1, h_2) \in \mathbb{T}^2.$$

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$$\int_{\mathbb{T}^2} \frac{d\mathbf{h}}{|\Phi(e^{i\mathbf{w}+\mathbf{h}})|^2} \ge C^{-1} \int_{\mathbb{T}^2} \frac{d\mathbf{h}}{||\mathbf{h}||^2} = \infty.$$

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Different properties for $d = 1$, $d = 2$ and $d > 2$.

Some general comments III

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Conjecture: $\mathbf{E}\log_+^d|Z_{\mathbf{t}}|<\infty$ is the necessary moment condition.

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Thank you for your attention!