Long-range dependence meets short-range dependence: multivariate limit theorems

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July 8, 2013

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Central limit theorem and short-range dependence

Classical Central Limit Theorem (CLT): $\{X(n)\}$ are *i.i.d.*, mean 0 and variance 1 (denoted as $X(n) \sim \mathrm{IID}(0,1)$),

$$\frac{1}{\sqrt{N}} \sum_{n=1}^{[Nt]} X(n) \Rightarrow B(t), \quad B(t)\text{: Brownian motion.}$$

CLT still holds if we replace i.i.d. with stationarity and some weak dependence (or say short-range dependence) condition, e.g., mixing, martingale difference.

Two important features about CLT:

- $\frac{1}{\sqrt{N}}$-normalization, or $\mathrm{Var}\left(\sum_{n=1}^{N} X(n)\right) \sim \sigma N$;
- Brownian motion limit.

What if $\{X_n\}$ is strongly dependent (or say long-range dependent)? Long-range dependence is usually characterized by a slow power decay of autocovariance:

$$\gamma(n) := \mathrm{Cov}(X_n, X_0) \sim c n^{\alpha},\ \alpha \in (-1, 0),$$

so $\sum_n |\gamma(n)| = \infty$.

Long-range dependent linear process

A common discrete-time stationary model for long-range dependence:

$$X(n) = \sum_{i=0}^{\infty} a_i \epsilon_{n-i}, \ \epsilon_i \sim \mathrm{IID}(0,1), \ a_n \sim c_0 n^{d-1}, \ d \in (0,1/2), \tag{1}$$

so that $\sum_i |a_i| = \infty$ but $\sum_i a_i^2 < \infty$. Then the autocovariance

$$\gamma(n) \sim c_1 n^{2d-1}, \quad 2d-1 \in (-1,0). \tag{2}$$

(2) implies that

$$\mathrm{Var}\left(\sum_{n=1}^{N} X(n)\right) \sim c_2 N^{2d+1},$$

or roughly equivalently the spectral density $f(\lambda) := \frac{1}{2\pi}\sum_n \gamma(n) e^{-in\lambda}$ satisfies

$$f(\lambda) \sim c_3 |\lambda|^{-2d}, \text{ as } \lambda \to 0.$$

The parameterization using d (called the *memory parameter*) is due to the fractional difference equation:

$$\Delta^d X(n) = \epsilon_n,$$

where $\Delta := I - B$ is the difference operator, and the stationary solution $X(n)$ can be written as a long-range dependent linear process as in (1).

Limit theorem for long-range dependent linear process

Theorem

[Davydov, 1970] Let $X(n)$ be as given in (1) ($a_n \sim cn^{d-1}, 0 < d < 1/2$). Then

$$\frac{1}{A(N)} \sum_{n=1}^{[Nt]} X(n) \Rightarrow B_H(t),$$

where B_H is the fractional Brownian motion with Hurst parameter $H = 1/2 + d$,

$$A(N) \sim \sqrt{\mathrm{Var}\left(\sum_{n=1}^{N} X(n)\right)} \sim c_2 N^H.$$

$B_H(t)$ is a 0-mean Gaussian process,
$\mathrm{Cov}(B_H(s), B_H(t)) = \frac{1}{2}\left(s^{2H} + t^{2H} - |s-t|^{2H}\right)$. It has stationary long-range dependent increments, and is H-self-similar: $B(at) \overset{f.d.d.}{=} a^H B(t)$, $\forall a > 0$.

$B_H(t)$ has a representation:

$$B_H(t) = \int_{\mathbb{R}} \int_0^t (s-x)_+^{d-1} ds \; W(dx), \quad W(.) : \text{Brownian random measure},$$

which hints why it is the limit of $\frac{1}{A(N)} \sum_{n=1}^{[Nt]} \sum_{i=0}^{\infty} a_i \epsilon_{n-i}$ where $a_n \sim cn^{d-1}$.

Multilinear polynomial-form process

Consider the following multilinear polynomial-form process:

$$X(n) = \sum_{0 \le i_1, \ldots, i_k < \infty}^{\prime} a_{i_1} \ldots a_{i_k} \epsilon_{n-i_1} \ldots \epsilon_{n-i_k}, \quad \epsilon_i \sim \mathrm{IID}(0,1), \tag{3}$$

where the prime $'$ indicates the exclusion of the diagonals $i_p = i_q$ for $p \neq q$, and k is the order. Such a process arises from study of non-linear functional of a linear process ([Surgailis, 1982]).

Definition

Set first $d_X = \frac{1}{2} - k(\frac{1}{2} - d)$. *Let* $X(n)$ *be given as in (3). We say*
$X(n)$ *is SRD, if* $|a_n| \le cn^{d-1}$ *and* $d_X < 0$ *(or equivalently* $d < \frac{1}{2}(1 - \frac{1}{k})$*);*
$X(n)$ *is LRD, if* $a_n \sim cn^{d-1}$ *and* $0 < d_X < 1/2$ *(or equivalently* $d > \frac{1}{2}(1 - \frac{1}{k})$*).*

Corollary

When $X(n)$ *is SRD,* $|\gamma(n)| \le c_1 n^{2d_X - 1}$, $\sum_n |\gamma(n)| < \infty$,
$\sqrt{\mathrm{Var}\left(\sum_{n=1}^{N} X(n)\right)} \sim \sigma N^{1/2}$;

When $X(n)$ *is LRD,* $\gamma(n) \sim c_1 n^{2d_X - 1}$, $\sum_n |\gamma(n)| = \infty$,
$\sqrt{\mathrm{Var}\left(\sum_{n=1}^{N} X(n)\right)} \sim c_2 N^{1/2 + d_X}$.

Limit theorem for a multilinear polynomial-form process $X(n)$

Let " $\stackrel{f.d.d.}{\longrightarrow}$ " denote convergence in finite-dimensional distributions. The following result can be found in [Giraitis et al., 2012]:

Theorem

If $X(n)$ is SRD, then

$$\frac{1}{A(N)}\sum_{n=1}^{[Nt]} X(n) \stackrel{f.d.d.}{\longrightarrow} B(t),$$

where $B(t)$ is the standard Brownian motion, $A(N) \sim \sigma\sqrt{N}$ with $\sigma^2 = \sum_n \gamma(n)$.

If $X(n)$ be LRD, then

$$\frac{1}{A(N)}\sum_{n=1}^{[Nt]} X(n) \stackrel{f.d.d.}{\longrightarrow} Z_d^{(k)}(t),$$

where $Z_d^{(k)}(t)$ is the so-called Hermite process (defined in the next slide), $A(N) \sim cN^{1/2+d_X}$ with $d_X = \frac{1}{2} - k(\frac{1}{2} - d) \in (0, \frac{1}{2})$.

Remark

The Hermite process $Z_d^{(k)}$ is fractional Brownian motion when $k = 1$. But when $k \geq 2$, it is non-Gaussian (belongs to higher-order Wiener chaos), although it shares the same covariance structure with fractional Brownian motion.

Hermite process $Z_d^{(k)}(t)$

The Hermite process $Z_d^{(k)}(t)$ appearing as the limit in the LRD case is defined with the aid of the multiple Wiener-Itô stochastic integral denoted by $I_k(.)$:

$$Z_d^{(k)}(t) = I_k(f_{k,d}^{(t)}) := \int_{\mathbb{R}^k}' f_{k,d}^{(t)}(x_1, \ldots, x_k) W(dx_1) \ldots W(dx_k) \tag{4}$$

where the prime $'$ indicates the exclusion of the diagonals $x_i = x_j$ for $i \neq j$, $W(.)$ is Brownian random measure, and

$$f_{k,d}^{(t)}(x_1, \ldots, x_k) = a_{k,d} \int_0^t (s - x_1)_+^{d-1} \ldots (s - x_k)_+^{d-1} ds, \tag{5}$$

for some constant $a_{k,d} > 0$.

Remark

The above representation of $Z_d^{(k)}(t)$ actually illustrates why it is the limit of

$$\frac{1}{A(N)} \sum_{n=1}^{[Nt]} \sum_{0 \leq i_1, \ldots, i_k < \infty}' a_{i_1} \ldots a_{i_k} \epsilon_{n-i_1} \ldots \epsilon_{n-i_k} :$$

$(s - x_1)_+^{d-1} \ldots (s - x_k)_+^{d-1}$ *corresponds to the product* $a_{i_1} \ldots a_{i_k}$ $(a_n \sim cn^{d-1})$.
$\int_0^t ds$ *is due to the sum* $\sum_{n=1}^{[Nt]}$.
$\int_{\mathbb{R}^k}' W(dx_1) \ldots W(dx_k)$ *comes from the off-diagonal aggregation of* $\epsilon_{i_1} \ldots \epsilon_{i_k}$.

Multivariate limit

The question studied in [Bai and Taqqu, 2013] is as follows:

Consider a vector of different multilinear polynomial-form process constructed on the same $\{\epsilon_i\}$, whose component is

$$X_j(n) := \sum_{0\le i_1,\ldots,i_{k_j}<\infty}' a_{i_1,j}\ldots a_{i_{k_j},j}\ \epsilon_{n-i_1}\ldots\epsilon_{n-i_{k_j}},\ j=1,\ldots,J,$$

where $k_1,\ldots,k_J$ are the orders for $X_1(n),\ldots,X_J(n)$ respectively. Let

$$Y_{j,N}(t) = \frac{1}{A_j(N)}\sum_{n=1}^{[Nt]} X_j(n),\ \ t\ge 0,$$

where $A_j(N)$ is a normalization factor such that $\lim_{N\to\infty}\mathrm{Var}\left(Y_{j,N}(1)\right)=1$.

What is the joint limit law of the vector

$$\mathbf{Y}_N(t) := \left(Y_{1,N}(t),\ldots,Y_{J,N}(t)\right)\ ?$$

Three cases: a) pure SRD case, b) pure LRD case and c) mixed SRD and LRD case.

Multivariate limit theorems: pure SRD case

Theorem

If all the components in $\mathbf{Y}_N$ are SRD, then

$$\mathbf{Y}_N(t) \xrightarrow{f.d.d.} \mathbf{B}(t) := (B_1(t), \ldots, B_J(t)),$$

where $B_1(t), \ldots, B_J(t)$ are standard Brownian motions with

$$\mathrm{Cov}\left(B_p(s), B_q(t)\right) = (s \wedge t)\frac{\sigma_{p,q}}{\sigma_p \sigma_q},$$

$$\sigma_p^2 = \sum_{n=-\infty}^{\infty} \gamma_p(n) := \sum_{n=-\infty}^{\infty} \mathrm{Cov}(X_p(n), X_p(0)),$$

$$\sigma_{p,q} = \sum_{n=-\infty}^{\infty} \gamma_{p,q}(n) := \sum_{n=-\infty}^{\infty} \mathrm{Cov}(X_p(n), X_q(0)).$$

The normalization $A_j(N) \sim \sigma_j \sqrt{N}$ as $N \to \infty$.

The theorem is proved by Crámer-Wold device and an approximation technique: truncate $X(n)$ at lag m, which results in $m-$dependent sequence, and let $m \to \infty$ at the end.

Multivariate limit theorems: pure LRD case

Theorem

If all the components in $\mathbf{Y}_N$ are LRD with $d = d_1, \dots d_J$ respectively, then

$$\mathbf{Y}_N(t) \xrightarrow{f.d.d.} \mathbf{Z}^{\mathbf{k}}_{\mathbf{d}}(t) = (Z^{(k_1)}_{d_1}(t), \dots, Z^{(k_J)}_{d_J}(t)),$$

where $Z^{(k_j)}_{d_j}(t)$ are Hermite processes sharing the same random measure $W(.)$ in their Wiener-Itô integral representations. The normalization $A_j(N) \sim c_j N^{1+(d_j-1/2)k_j}$ as $N \to \infty$ for some $c_j > 0$.
Moreover, the processes $Z^{(k_j)}_{d_j}$, $j = 1, \dots, J$ are dependent.

The theorem is proved by a multivariate extension of the univariate result concerning the convergence of a polynomial-form (discrete chaos) to a Wiener-Itô integral:

$$Q(h_N) := \sum_{i_1,\dots,i_k}{}' h_N(i_1, \dots, i_k)\epsilon_{i_1} \dots \epsilon_{i_k}$$
$$\xrightarrow{d} \int_{\mathbb{R}^k}' f(x_1, \dots, x_k) W(dx_1) \dots W(dx_k) =: I(f)$$

as $N \to \infty$.

Dependence between LRD limit components

Now we show why in the limit the Hermite processes of different orders are dependent.

From [Ustunel and Zakai, 1989], we have the following criterion for the independence of multiple Wiener-Itô integrals: suppose that symmetric $g_1 \in L^2(\mathbb{R}^p)$ and $g_2 \in L^2(\mathbb{R}^q)$. Then multiple integrals $I_p(g_1)$ and $I_q(g_2)$ $(p, q \geq 1)$ which share the same random measure are independent if and only if

$$g_1 \otimes_1 g_2 := \int_{\mathbb{R}} g_1(x_1, \ldots, x_{p-1}, u) g_2(x_p, \ldots, x_{p+q-2}, u) du = 0 \quad a.s..$$

One can apply this to the Hermite processes:

$$\begin{aligned} &(g_{p,d} \otimes_1 g_{q,d})(x_1, \ldots, x_{p+q-2}) \\ = &\int_{\mathbb{R}} \left(\int_0^t \prod_{j=1}^{p-1} (s - x_j)_+^{d-1} (s-u)_+^{d-1} ds \int_0^t \prod_{j=p}^{p+q-2} (s - x_j)_+^{d-1} (s-u)_+^{d-1} ds \right) du > 0 \end{aligned}$$

since everything involved in the integrand is positive.

Multivariate limit theorems: mixed SRD and LRD case

Theorem

Break $\mathbf{Y}_N$ *into 3 parts:*

$$\mathbf{Y}_N = (\mathbf{Y}_{N,S_1}, \mathbf{Y}_{N,S_2}, \mathbf{Y}_{N,L}),$$

$\mathbf{Y}_{N,S_1}$*:* J_{S_1}*−dimensional, every component is SRD, order* $k_{j,S_1} = 1$*,*
$\mathbf{Y}_{N,S_2}$*:* J_{S_2}*−dimensional, every component is SRD, order* $k_{j,S_2} \geq 2$*,*
$\mathbf{Y}_{N,L}$*:* J_L*−dimensional, every component is LRD.*
Then

$$\mathbf{Y}_N(t) = (\mathbf{Y}_{N,S_1}(t), \mathbf{Y}_{N,S_2}(t), \mathbf{Y}_{N,L}(t)) \xrightarrow{f.d.d.} (\mathbf{W}(t), \mathbf{B}(t), \mathbf{Z}_{\mathbf{d}_L}^{\mathbf{k}_L}(t)), \qquad (6)$$

$\mathbf{B}(t)$*: the multivariate Gaussian process appearing in the pure SRD case,*
$\mathbf{Z}_{\mathbf{d}_L}^{\mathbf{k}_L}(t)$*: the multivariate Hermite process appearing in the pure LRD case,*
$\mathbf{W}(t) := (W(t), \ldots, W(t))$*, where* $W(t)$ *is the Brownian motion integrator for representing* $\mathbf{Z}_{\mathbf{d}_L}^{\mathbf{k}_L}(t)$*.*
Moreover, $\mathbf{B}(t)$ *is independent of* $(\mathbf{W}(t), \mathbf{Z}_{\mathbf{d}_L}^{\mathbf{k}_L}(t))$*.*

The heuristic reason why $\mathbf{B}(t)$ is independent of $(\mathbf{W}(t), \mathbf{Z}_{\mathbf{d}_L}^{\mathbf{k}_L}(t))$:
$\mathbf{Y}_{N,S_2}(t)$ belongs to chaos of order ≥ 2,
$\mathbf{Y}_{N,S_1}(t)$ belongs to chaos of order 1,
$\{\epsilon_i\}$ belongs to chaos of order 1, and after summing becomes asymptotically the Brownian measure $W(.)$ representing $\mathbf{Z}_{\mathbf{d}_L}^{\mathbf{k}_L}(t)$.

Summary and remarks

1. Long-range dependence typically results in limits other than Brownian motion, and when it meets non-linearity, the limit may not even be Gaussian.

2. We obtain the multivariate limit for a vector made up of different multilinear polynomial-form processes defined through the same $\{\epsilon_i\}$ with possibly a mixture of SRD and LRD. An interesting asymptotic independence feature is observed in the mixed limit.

3. Under some additional assumptions on the SRD components, the multivariate convergence in finite-dimensional distributions extend to weak convergence in $D[0,1]^J$.

4. In another paper [Bai and Taqqu, 2012], the same multivariate limit problem is studied in the context of nonlinear functions of long-range dependent Gaussian process. In this case, the moment independence is established by a result of [Nourdin and Rosinski, 2012], but the distributional independence remains an open problem in general, except when the Gaussian process is linear with regularly varying weights.

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Thank you!