A transience condition for a class of one-dimensional symmetric Lévy processes

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Outline

Motivation

An $\mathbb{R}^d$-valued, $d \geq 1$, Lévy process $\{L_t\}_{t\geq 0}$ defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is said to be transient if

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or, if and only if

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Similarly, a Lévy process $\{L_t\}_{t\geq 0}$ is recurrent if and only if

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Similarly, a Lévy process $\{L_t\}_{t\geq 0}$ is recurrent if and only if

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Problem: The above characterizations (definitions) are not operable.

Every Lévy process $\{L_t\}_{t\geq 0}$ can be described in terms of its Fourier transform,

$$\mathbb{E}\left[e^{i\langle \xi, L_t\rangle}\right] = e^{-t\psi(\xi)}, \quad t \geq 0,\ \xi \in \mathbb{R}^d,$$

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$$\mathbb{E}\left[e^{i\langle\xi,L_t\rangle}\right] = e^{-t\psi(\xi)}, \quad t \geq 0,\ \xi \in \mathbb{R}^d,$$

where the characteristic exponent $\psi : \mathbb{R}^d \longrightarrow \mathbb{C}$ has the following representation

$$\psi(\xi) = i\langle\xi, b\rangle + \frac{1}{2}\langle\xi, c\xi\rangle + \int_{\mathbb{R}^d}\left(1 - e^{i\langle\xi,y\rangle} + i\langle\xi, y\rangle 1_{\{|y|\leq 1\}}\right)\nu(dy).$$

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Chung-Fuchs criterion: A Lévy process $\{L_t\}_{t\geq 0}$ is transient if and only if

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Aim: To find conditions for the transience and recurrence property of Lévy processes in terms of the Lévy triplet (b, c, ν).

Theorem 1 (L. A. Shepp 1962)

Let $\{L_t\}_{t\geq 0}$ be a one-dimensional symmetric Lévy process with the Lévy measure $\nu(dy)$. Then, $\{L_t\}_{t\geq 0}$ is recurrent if

$$\int_1^\infty \left(\int_0^y z\nu\left(\max\{1, z\}, \infty\right) dz\right)^{-1} dy = \infty. \quad (1)$$

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Theorem 2 (N. S. 2013)

Let $\{L_t\}_{t\geq 0}$ be a one-dimensional symmetric Lévy process with the Lévy measure $\nu(dy) = f(y)dy$ or $\nu(n) = p_n$, where $f(y) > 0$ a.e. and $p_n > 0$ for all $n \geq 1$. Then, $\{L_t\}_{t\geq 0}$ is transient if

$$\int_1^\infty \frac{dy}{y^3 f(y)} < \infty \quad \text{or} \quad \sum_{n=1}^\infty \frac{1}{n^3 p_n} < \infty. \tag{2}$$

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A one-dimensional symmetric stable Lévy process or a random walk is transient if $\alpha < 1$.

- An example where the condition (2) is more suitable than the Chung-Fuchs criterion is obtained by combining two "indices of stability". Let $\{L_t\}_{t\geq 0}$ be a one-dimensional symmetric Lévy process with the Lévy measure $\nu(n) = p_n$, where $p_{2n} = (2n)^{-\alpha-1}$ and $p_{2n-1} = (2n-1)^{-\beta-1}$ for $n \geq 1$ and $\alpha, \beta \in (0, 2)$.

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$$\sum_{n=1}^{\infty} \frac{1}{n^3 p_n} < \infty. \tag{3}$$

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Step 2. In the second step, we prove that a one-dimensional symmetric random walk $\{S_n\}_{n\geq 0}$ with continuous jumps $\mathbb{P}(S_1 \in dy) = f(y)dy$, where $f(y) > 0$ a.e., is transient if

$$\int_1^{\infty} \frac{1}{y^3 f(y)} < \infty. \tag{4}$$

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More precisely, for $\delta > 0$ we define a discretization of $\{S_n\}_{n\geq 0}$ as a random walk $\{S_n^\delta\}_{n\geq 0}$ on $\delta\mathbb{Z}$ with jumps

$$\mathbb{P}(S_1^\delta = \delta n) := \int_{\delta n - \frac{\delta}{2}}^{\delta n + \frac{\delta}{2}} f(y)dy, \quad n \in \mathbb{Z}.$$

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- for $p_n := \int_{n-\frac{1}{2}}^{n+\frac{1}{2}} f(y)dy$, $n \in \mathbb{Z}$, the condition (4) implies the condition (3), that is, by Step 1, the condition (4) implies the transience property of $\{S_n^1\}_{n\geq 0}$.

Step 3. In the third step, we consider the Lévy process case. Recall that $\{L_t\}_{t\geq 0}$ is a one-dimensional symmetric Lévy process with the Lévy measure $\nu(dy) = f(y)dy$ or $\nu(n) = p_n$, where $f(y) > 0$ a.e. and $p_n > 0$ for all $n \geq 1$.

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In order to make a connection with the random walk case, we need the following result.

Theorem 3 (L. A. Shepp 1962)

Let $\{L_t^1\}_{t\geq 0}$ and $\{L_t^2\}_{t\geq 0}$ be one-dimensional symmetric Lévy processes with the Lévy measures $\nu_1(dy)$ and $\nu_2(dy)$, respectively. If $\nu_1(y,\infty) = \nu_2(y,\infty)$ for all $y \geq y_0$, for some $y_0 > 0$, then $\{L_t^1\}_{t\geq 0}$ and $\{L_t^2\}_{t\geq 0}$ are either both transient or both recurrent.

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According to Theorem 3, it suffices to consider the compound process case, that is,

$$\{L_t\}_{t\geq 0} \stackrel{\mathrm{d}}{=} \{S_{P_t}\}_{t\geq 0},$$

where $\{S_n\}_{n\geq 0}$ is a random walk with jumps $\mathbb{P}(S_1 \in dy) := \frac{1}{\nu(\mathbb{R})} f(y)dy$ (resp. $\mathbb{P}(S_1 = n) := \frac{1}{\nu(\mathbb{Z})} p_n$) and $\{P_t\}_{t\geq 0}$ is the Poisson process with parameter $\nu(\mathbb{R})$ (resp. $\nu(\mathbb{Z})$) independent of $\{S_n\}_{n\geq 0}$.

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Now, the desired result follows from Step 1 and Step 2.

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Thank you for your attention!