Unavoidable collections of balls for isotropic Lévy processes

Ante Mimica
(joint work with Zoran Vondraček)

Department of Mathematics
University of Zagreb
Croatia

Będlewo, July 9, 2013
\[
\left\{ \overline{B}(x_n, r_n) : n \in \mathbb{N} \right\} \quad \text{family of closed balls in } \mathbb{R}^d
\]

\[
A = \bigcup_{n=1}^{\infty} \overline{B}(x_n, r_n) \quad \text{'bubbles'}
\]

**Avoidability**

A is **avoidable** for a transient Markov process \( X = \{X_t\}_{t \geq 0} \) if

\[
P_0( T_A < \infty ) < 1,
\]

where \( T_A = \inf\{ t > 0 : X_t \in A \} \).

A is **unavoidable** if \( P_0( T_A < \infty ) = 1 \)
Brownian motion in $\mathbb{R}^d$

- $d = 1, 2$: any ball is unavoidable
- enough to consider transient BM, i.e. $d \geq 3$
Motivation

Theorem (Gardiner/Ghergu 2010)

Let $X$ be a Brownian motion in $\mathbb{R}^d$ with $d \geq 3$.

(i) If $A$ is unavoidable, then

$$\sum_{n=1}^{\infty} \left( \frac{r_n}{|x_n|} \right)^{d-2} = \infty.$$  \hspace{1cm} (\star)

(ii) Conversely, if (\star) and the separation condition

$$\inf_{j \neq k} \frac{|x_j - x_k|^d}{r_k^{d-2}|x_k|^2} > 0$$

hold, then $A$ is unavoidable.
Newtonian potential \((d \geq 3)\)

\[
G(x) = \frac{4\pi^{d/2}}{\Gamma\left(\frac{d}{2} - 1\right)} |x|^{2-d}, \quad x \neq 0
\]

For \(x \in \overline{B}(x_0, r)^c\)

\[
\mathbb{P}_x(T_{\overline{B}(x_0, r)} < \infty) = \frac{r^{d-2}}{|x - x_0|^{d-2}} = \frac{G(|x - x_0|)}{G(r)}
\]

Note:

\[
\sum_{n=1}^{\infty} \left(\frac{r_n}{|x_n|}\right)^{d-2} = \sum_{n=1}^{\infty} \frac{G(|x_n|)}{G(r_n)} = \sum_{n=1}^{\infty} \mathbb{P}_0(T_{\overline{B}(x_n, r_n)} < \infty)
\]
Regularly located balls

A family of balls \( \{ \overline{B}(x_n, r_n) : n \in \mathbb{N} \} \) is regularly located if:

(i) \( \exists \, \varepsilon > 0 \) such that \( |x_n - x_m| > 2\varepsilon, \ m \neq n \)

(ii) \( \exists \, R > 0 \) such that

\[
\forall x \in \mathbb{R}^d \quad B(x, R) \cap \{ x_n : n \in \mathbb{N} \} \neq \emptyset
\]

(iii) \( \exists \, \phi : (0, \infty) \to (0, \infty) \) decreasing such that

\[
r_n = \phi(|x_n|), \ n \in \mathbb{N}.
\]

Theorem (Carroll/Ortega-Cerdà 2006)

Let \( \{ \overline{B}(x_n, r_n) : n \in \mathbb{N} \} \) be a regularly located collection of balls. Then this collection is avoidable if and only if

\[
\int_{1}^{\infty} r \phi(r)^{d-2} \, dr < \infty.
\]
Setting

Assumptions

Let $X = \{X_t\}_{t \geq 0}$ be a transient Lévy process. Assume that

(a) it is isotropic unimodal, i.e. $\mathbb{P}_0(X_t \in dy) = p_t(|y|) dy$ for a decreasing function $p_t : \mathbb{R}^d \to [0, \infty)$

(b) the Lévy exponent $\psi$ satisfies the weak lower scaling condition

$$\psi(\lambda \xi) \geq c_L \lambda^\alpha \psi(\xi), \quad \lambda \geq 1, \ \xi \in \mathbb{R}^d \quad \text{(WLSC)}$$

for some $\alpha \in (0, 2]$ and $c_L \in (0, 1)$
Green potential of $X$ exists (transience) and is isotropic:

$$G(x) = \int_0^\infty p_t(|x|) \, dt =: g(|x|)$$

$g$ is decreasing

- $\psi$ is isotropic:
  $$\psi(\xi) = \psi_0(|\xi|)$$

  ($\psi_0$ is not necessarily increasing!).

Set

$$\psi_0^*(r) := \sup_{s \leq r} \psi_0(s).$$

(cf. Grzywny 2013)
Theorem (M/Vondraček 2013)

Let $X$ be an isotropic Lévy process in $\mathbb{R}^d$ with $d \geq 3$ satisfying the weak lower scaling condition.

(i) If $A$ is unavoidable, then

$$\sum_{n=1}^{\infty} \frac{G(|x_n|)}{G(r_n)} = \infty. \quad (\star)$$

(ii) Conversely, if $(\star)$ and the separation condition

$$\inf_{j \neq k} |x_j - x_k|^d \psi_0^*(|x_k|^{-1}) G(r_k) > 0$$

hold, then $A$ is unavoidable.
Isotropic stable processes: \( \psi(\xi) = |\xi|^{\alpha}, \ \alpha \in (0, 2) \)

\[
G(x) = \frac{2^\alpha \pi^{d/2}}{\Gamma\left(\frac{d}{2} - \frac{\alpha}{2}\right)} |x|^{2-\alpha}
\]

Riesz potential

\[
\mathbb{P}_x\left( T_{\overline{B}(x_0, r)} < \infty \right) \asymp \frac{r^{d-\alpha}}{|x-x_0|^{d-\alpha}}, \quad x \in \overline{B}(x_0, r)^c
\]

Separation condition reads:

\[
\inf_{j \neq k} \frac{|x_j - x_k|^d}{r_k^{d-\alpha} |x_k|^{\alpha}} > 0 \quad (\star)
\]

Under (\star):

\[
A \text{ is avoidable} \iff \sum_{n=1}^{\infty} \left( \frac{r_n}{|x_n|} \right)^{d-\alpha} < \infty.
\]
In the case of regularly located balls with \( r_n = \phi(|x_n|) \) the following holds:

**Theorem (M/Vondraček 2013)**

Let \( \{ \overline{B}(x_n, r_n) : n \in \mathbb{N} \} \) be a regularly located collection of balls. Then this collection is avoidable if and only if

\[
\int_1^\infty r^{d-1} \frac{G(r)}{G(\phi(r))} \, dr < \infty.
\]
Poissonian collection of balls

Consider a Poisson point process with mean measure $\mu(x) \, dx$.

$\phi$ radius function

Assumptions:

- $\mu(y) \asymp \mu(x)$, $\phi(y) \asymp \phi(x)$, $y \in B(x, |x|/2)$
- $\phi(x) \leq |x|/2$
- $|x|^2 G(\phi(x))^{-1} \mu(x) \leq C$

$X = \{ X_t, \mathbb{P}_x \}_{t \geq 0, x \in \mathbb{R}^d}$ independent subordinate Brownian motion satisfying (WLSC)
realization of points from PPP

A random collection of balls

\[ A_\mathcal{P} = \bigcup_{x \in \mathcal{P}} \overline{B}(x, \phi(|x|)) \]

We say that \( A_\mathcal{P} \) is avoidable if there exists \( x \in \mathbb{R}^d \) s.t.

\[ \mathbb{P}_x(T_{A_\mathcal{P}} < \infty) < 1. \]

Percolation Lévy process occurs if there is a positive probability that the realization of points from the Poisson point process results in avoidable collection of balls.
Theorem (M/Vondraček 2013)

Percolation Lévy process occurs if and only if

\[ \int_{|x|>1} \frac{G(x)}{G(\phi(x))} \mu(x) \, dx < \infty. \]

Moreover, in case percolation Lévy process occurs, the random collection of balls \( A_P \) is avoidable with probability 1.
Analytic approach

$A \subset \mathbb{R}^d$ is minimally thin at infinity if $\mathbb{P}_0(T_A < \infty) < 1$.

Wiener-type criterion

Let $\{Q_n : n \in \mathbb{N}\}$ be the Whitney decomposition of $\mathbb{R}^d \setminus \{0\}$. Then $A \subset \mathbb{R}^d$ is minimally thin with respect to the Brownian motion if and only if

$$\sum_{n=1}^{\infty} (\text{diam}(Q_n))^{d-2} \text{Cap}(A \cap Q_n) < \infty.$$ 

M/Vondraček 2013

A similar criterion holds for the process $X$:

$$\sum_{n=1}^{\infty} G(\text{diam}(Q_n)) \text{Cap}(A \cap Q_n) < \infty.$$
Thank you for your attention!


