Unavoidable collections of balls for isotropic Lévy processes

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$\{\overline{B}(x_n, r_n)\colon\ n \in \mathbb{N}\}$ family of closed balls in $\mathbb{R}^d$

$A = \bigcup_{n=1}^{\infty} \overline{B}(x_n, r_n)$ 'bubbles'

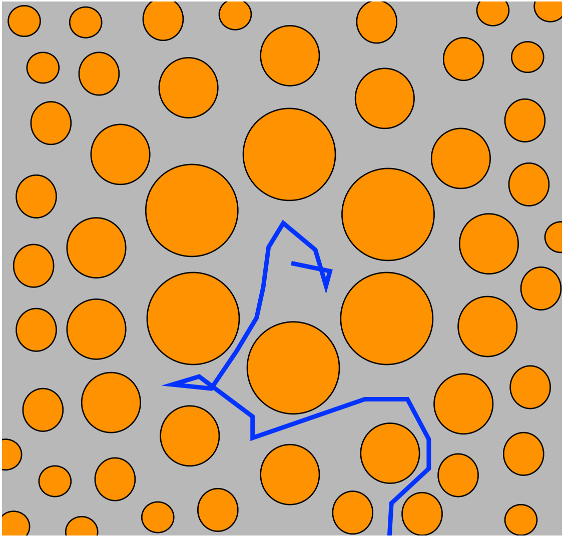
Avoidability

A is avoidable for a transient Markov process $X = \{X_t\}_{t\geq 0}$ if

$$\mathbb{P}_0(T_A < \infty) < 1,$$

where $T_A = \inf\{t > 0\colon X_t \in A\}$.

A is unavoidable if $\mathbb{P}_0(T_A < \infty) = 1$



X Brownian motion in $\mathbb{R}^d$

- $d = 1, 2$: any ball is unavoidable
- enough to consider transient BM, i.e. $d \geq 3$

Motivation

Theorem (Gardiner/Ghergu 2010)

Let X be a Brownian motion in $\mathbb{R}^d$ with $d \geq 3$.

(i) *If A is unavoidable, then*

$$\sum_{n=1}^{\infty} \left(\frac{r_n}{|x_n|} \right)^{d-2} = \infty. \qquad (\star)$$

(ii) *Conversely, if $(\star)$ and the separation condition*

$$\inf_{j \neq k} \frac{|x_j - x_k|^d}{r_k^{d-2} |x_k|^2} > 0$$

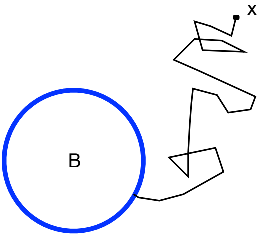
hold, then A is unavoidable.

Newtonian potential ($d \geq 3$)

$$G(x) = \frac{4\pi^{d/2}}{\Gamma(\frac{d}{2}-1)}|x|^{2-d}, \quad x \neq 0$$

For $x \in \overline{B}(x_0, r)^c$

$$\mathbb{P}_x(T_{\overline{B}(x_0,r)} < \infty) = \frac{r^{d-2}}{|x-x_0|^{d-2}} = \frac{G(|x-x_0|)}{G(r)}$$



Note:

$$\sum_{n=1}^{\infty}\left(\frac{r_n}{|x_n|}\right)^{d-2} = \sum_{n=1}^{\infty}\frac{G(|x_n|)}{G(r_n)} = \sum_{n=1}^{\infty}\mathbb{P}_0(T_{\overline{B}(x_n,r_n)} < \infty)$$

Regularly located balls

A family of balls $\{\overline{B}(x_n, r_n) \colon n \in \mathbb{N}\}$ is regularly located if:

(i) $\exists\ \varepsilon > 0$ such that $|x_n - x_m| > 2\varepsilon, \quad m \neq n$

(ii) $\exists\ R > 0$ such that

$$\forall x \in \mathbb{R}^d \qquad B(x, R) \cap \{x_n \colon n \in \mathbb{N}\} \neq \emptyset$$

(iii) $\exists\ \phi \colon (0, \infty) \to (0, \infty)$ decreasing such that

$$r_n = \phi(|x_n|), n \in \mathbb{N}.$$

Theorem (Carroll/Ortega-Cerdà 2006)

Let $\{\overline{B}(x_n, r_n) \colon n \in \mathbb{N}\}$ be a regularly located collection of balls. Then this collection is avoidable if and only if

$$\int_1^\infty r\phi(r)^{d-2}\, dr < \infty.$$

Setting

Assumptions

Let $X = \{X_t\}_{t\geq 0}$ be a transient Lévy process. Assume that

(a) it is isotropic unimodal, i.e. $\mathbb{P}_0(X_t \in dy) = p_t(|y|)\,dy$ for a decreasing function $p_t : \mathbb{R}^d \to [0,\infty)$

(b) the Lévy exponent ψ satisfies the weak lower scaling condition

$$\psi(\lambda\xi) \geq c_L \lambda^\alpha \psi(\xi), \qquad \lambda \geq 1,\ \xi \in \mathbb{R}^d \qquad \text{(WLSC)}$$

for some $\alpha \in (0,2]$ and $c_L \in (0,1)$

- Green potential of X exists (transience) and is isotropic:

$$G(x) = \int_0^\infty p_t(|x|)\,dt =: g(|x|)$$

g is decreasing

- ψ is isotropic:

$$\psi(\xi) = \psi_0(|\xi|)$$

(ψ_0 is not necessarily increasing!).

Set

$$\psi_0^\star(r) := \sup_{s \le r} \psi_0(s).$$

(cf. Grzywny 2013)

Results

Theorem (M/Vondraček 2013)

Let X be an isotropic Lévy process in $\mathbb{R}^d$ with $d \geq 3$ satisfying the weak lower scaling condition.

(i) *If A is unavoidable, then*

$$\sum_{n=1}^{\infty} \frac{G(|x_n|)}{G(r_n)} = \infty. \qquad (\star)$$

(ii) *Conversely, if $(\star)$ and the separation condition*

$$\inf_{j \neq k} |x_j - x_k|^d \psi_0^\star(|x_k|^{-1}) G(r_k) > 0$$

hold, then A is unavoidable.

Isotropic stable processes: $\psi(\xi) = |\xi|^\alpha$, $\alpha \in (0,2)$

$$G(x) = \frac{2^\alpha \pi^{d/2}}{\Gamma(\frac{d}{2} - \frac{\alpha}{2})} |x|^{2-\alpha}$$ Riesz potential

$$\mathbb{P}_x(T_{\overline{B}(x_0,r)} < \infty) \asymp \frac{r^{d-\alpha}}{|x - x_0|^{d-\alpha}}, \quad x \in \overline{B}(x_0, r)^c$$

Separation condition reads:

$$\inf_{j \neq k} \frac{|x_j - x_k|^d}{r_k^{d-\alpha} |x_k|^\alpha} > 0 \qquad (\star)$$

Under $(\star)$:

$$A \text{ is avodiable} \iff \sum_{n=1}^{\infty} \left(\frac{r_n}{|x_n|} \right)^{d-\alpha} < \infty.$$

In the case of regularly located balls with $r_n = \phi(|x_n|)$ the following holds:

Theorem (M/Vondraček 2013)

Let $\{\overline{B}(x_n, r_n) \colon n \in \mathbb{N}\}$ be a regularly located collection of balls. Then this collection is avoidable if and only if

$$\int_1^\infty r^{d-1} \frac{G(r)}{G(\phi(r))} \, dr < \infty .$$

Poissonian collection of balls

Consider a Poisson point process with mean measure $\mu(x)\,dx$.

ϕ radius function

Assumptions:

- $\mu(y) \asymp \mu(x)$, $\phi(y) \asymp \phi(x)$, $y \in B(x, |x|/2)$
- $\phi(x) \le |x|/2$
- $|x|^2 G(\phi(x))^{-1} \mu(x) \le C$

$X = \{X_t, \mathbb{P}_x\}_{t\ge 0, x\in\mathbb{R}^d}$ independent subordinate Brownian motion satisfying (WLSC)

$\mathscr{P}$ realization of points from PPP

Random collection of balls

$$A_{\mathscr{P}} = \bigcup_{x \in \mathscr{P}} \overline{B}(x, \phi(|x|))$$

We say that $A_{\mathscr{P}}$ is avoidable if there exists $x \in \mathbb{R}^d$ s.t.

$$\mathbb{P}_x(T_{A_{\mathscr{P}}} < \infty) < 1\,.$$

Percolation Lévy process occurs if there is a positive probability that the realization of points from the Poisson point process results in avoidable collection of balls.

Theorem (M/Vondraček 2013)

Percolation Lévy process occurs if and only if

$$\int_{|x|>1} \frac{G(x)}{G(\phi(x))}\mu(x)\,dx < \infty.$$

Moreover, in case percolation Lévy proces occurs, the random collection of balls $A_{\mathscr{P}}$ *is avoidable with probability 1.*

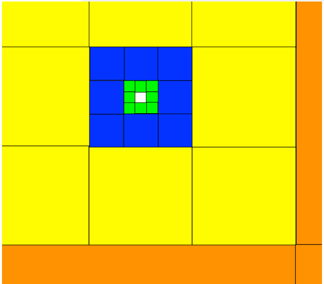
Analytic approach

$A \subset \mathbb{R}^d$ is minimally thin at infinity if $\mathbb{P}_0(T_A < \infty) < 1$.

Wiener-type criterion

Let $\{Q_n \colon n \in \mathbb{N}\}$ be the Whitney decomposition of $\mathbb{R}^d \setminus \{0\}$. Then $A \subset \mathbb{R}^d$ is minimally thin with respect to the Brownian motion if and only if

$$\sum_{n=1}^{\infty} (\mathrm{diam}(Q_n))^{d-2} \mathrm{Cap}(A \cap Q_n) < \infty .$$



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A similar criterion holds for the process X:

$$\sum_{n=1}^{\infty} G(\mathrm{diam}(Q_n)) \mathrm{Cap}(A \cap Q_n) < \infty .$$

Thank you for your attention!

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