A strong law of large numbers for the fragmentation energy model

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Outline

Let $\mathcal{P}$ denote the space of partitions $\pi := (\pi_n)_{n\in\mathbb{N}}$ of $\mathbb{N}$, ordered such that

$$\forall\, i \le j \in \mathbb{N}: \quad \inf \pi_i \le \inf \pi_j,$$

where $\inf \emptyset := \infty$.

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Definition 1.1

We call a $\mathcal{P}$–valued Markov process $\Pi := (\Pi(t))_{t\in\mathbb{R}_0^+}$, which is continuous in probability, *homogeneous $\mathcal{P}$–fragmentation process* if

(i) $\Pi(0) = (\mathbb{N}, \emptyset, \ldots)$,

(ii) for any $s, t \in \mathbb{R}_0^+$ we have

$$\mathbb{P}\left(\Pi(s+t) \in (\cdot) \middle| \, \Pi(s)\right) = \mathbb{P}\left(\left(\Pi^{(n)}(t)\Big|_{\pi_n}\right)^*_{n\in\mathbb{N}} \in (\cdot)\right)\Bigg|_{\pi=\Pi(s)},$$

where the $\Pi^{(n)}$ are i.i.d. copies of Π and
where $(\cdot)^*$ denotes the reordering to obtain an element of $\mathcal{P}$.

Bertoin (2001): Homogeneous fragmentations are characterised by

- a nonnegative rate of erosion,
- a σ–finite measure ν, called *dislocation measure*, on

$$\mathcal{S} := \left\{ \mathbf{s} := (s_n)_{n\in\mathbb{N}} : s_1 \geq s_2 \ldots \geq 0, \sum_{n\in\mathbb{N}} s_n \leq 1 \right\},$$

which satisfies

$$\nu(\{1,0,\ldots\}) = 0 \quad \text{and} \quad \int_{\mathcal{S}} (1-s_1)\nu(\mathrm{d}\mathbf{s}) < \infty.$$

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- no erosion,
- $\nu(\mathcal{S}) = \infty$ allowed,
- $\nu\left(\mathbf{s} \in \mathcal{S} : \sum_{n\in\mathbb{N}} s_n < 1\right) > 0$ allowed.

Consider the exchangeable partition measure μ on $\mathcal{P}$ given by

$$\mu(\mathrm{d}\pi) = \int_{\mathcal{S}} \varrho_{\mathbf{s}}(\mathrm{d}\pi)\nu(\mathrm{d}\mathbf{s}),$$

where $\varrho_{\mathbf{s}}$ is the law of Kingman's paint-box based on $\mathbf{s} \in \mathcal{S}$.

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Let $(\pi(t), k(t))_{t\in\mathbb{R}_0^+}$ be a $\mathcal{P} \times \mathbb{N}$–valued Poisson point process with characteristic measure $\mu \otimes \sharp$, where $\sharp$ is the counting measure on $\mathbb{N}$.

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Then

- Π changes state only at the times $(t_i)_{i \in \mathcal{I}}$ at which an atom $(\pi(t_i), k(t_i))$ occurs in $\mathcal{P} \setminus \{(\mathbb{N}, \emptyset, \cdots)\} \times \mathbb{N}$.

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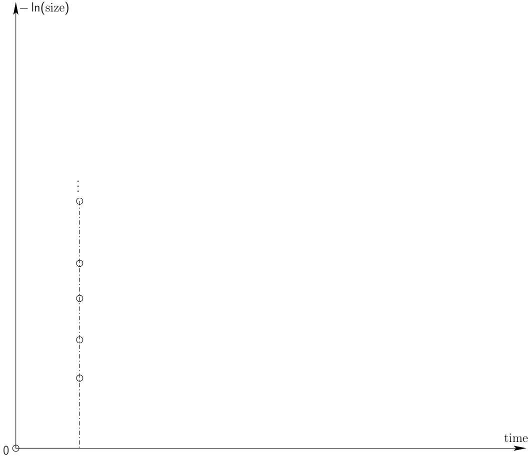
- At any time t_i the sequence $\Pi(t_i)$ is obtained from $\Pi(t_i-)$ by partitioning the $k(t_i)$–th block into the sub–blocks

$$(\Pi_{k(t_i)}(t_i-) \cap \pi_n(t_i))_{n\in\mathbb{N}}.$$

Figure: Illustration of Π.



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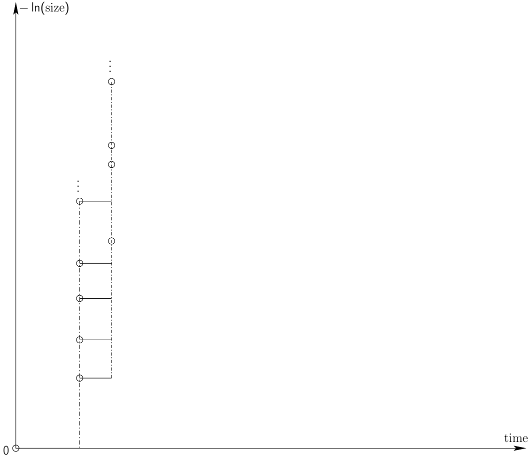

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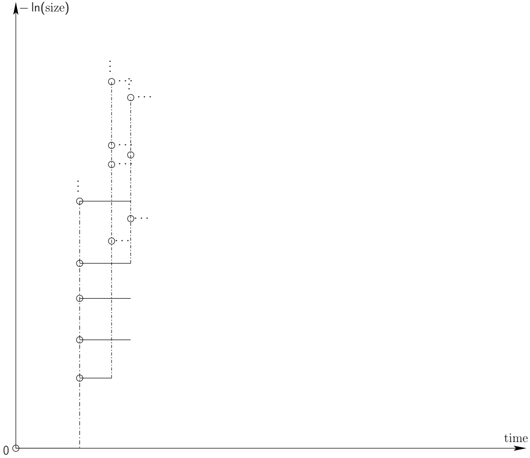

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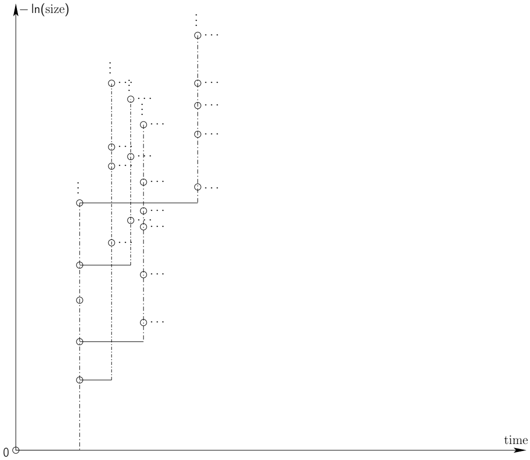

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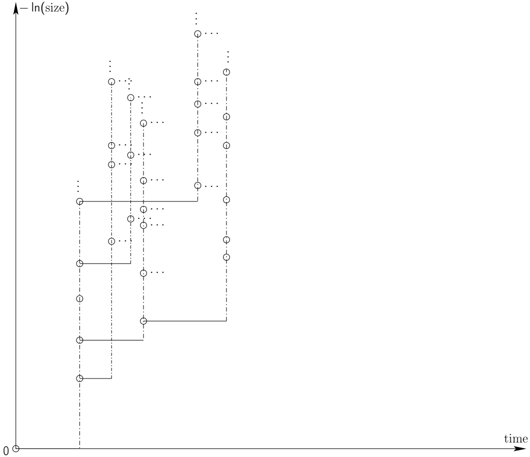

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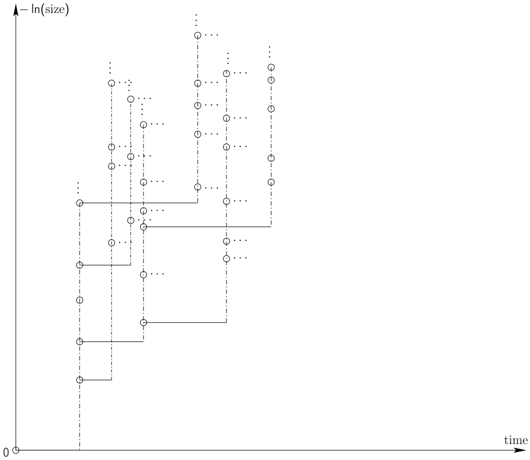

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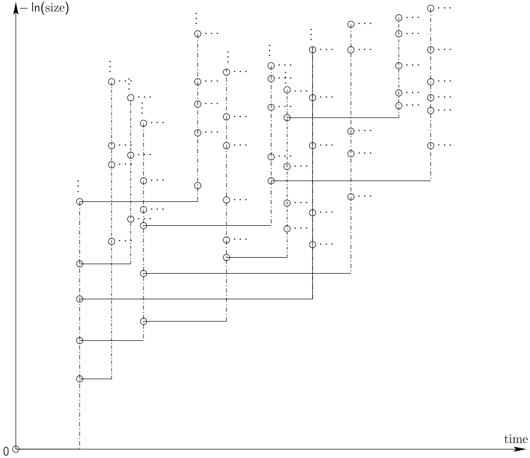

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Set

$$\underline{p} := \inf \left\{ p \in \mathbb{R} : \int_{\mathcal{S}} \left| 1 - \sum_{n \in \mathbb{N}} s_n^{1+p} \right| \nu(ds) < \infty \right\} \in [-1, 0].$$

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The function $\Phi : (\underline{p}, \infty) \to \mathbb{R}$, given by

$$\forall\, p \in (\underline{p}, \infty) : \quad \Phi(p) = \int_{\mathcal{S}} \left(1 - \sum_{n \in \mathbb{N}} s_n^{1+p} \right) \nu(\mathrm{d}s),$$

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Hypothesis: There exists a $p^* \in (\underline{p}, 0]$ such that

$$\Phi(p^*) = 0.$$

For every $\eta \in (0,1]$ we denote by $(\lambda_{\eta,k})_{k\in\mathbb{N}}$ the terminal sizes of the fragmentation stopped at the stopping line that corresponds to the first blocks, in their respective "line of descent", of size less than η.

Illustration of $(\lambda_{\eta,k})_{k\in\mathbb{N}}$.

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▸ Illustration of $(\lambda_{\eta,k})_{k\in\mathbb{N}}$.

Consider the process $(\Lambda_\eta(p^*))_{\eta\in(0,1]}$, defined by

$$\Lambda_\eta(p^*) := \sum_{k\in\mathbb{N}} \lambda_{\eta,k}^{1+p^*}$$

for each $\eta \in (0,1]$.

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It can be shown that $(\Lambda_\eta(p^*))_{\eta\in(0,1]}$ is a uniformly integrable unit–mean martingale. We shall be interested in its a.s. limit

$$\Lambda_0(p^*) := \lim_{\eta\downarrow 0} \Lambda_\eta(p^*).$$

Consider a measurable random function $\psi : \mathcal{P} \times \Omega \to \mathbb{R}_0^+$ that is independent of Π. Further, define the *energy cost* $\mathcal{E}_p$, $p < p^*$, by

$$\mathcal{E}_p(\eta) = \sum_{i \in \mathcal{I}} \mathbb{1}_{\{|\Pi_{\kappa(t_i)}(t_i-)| \geq \eta\}} |\Pi_{\kappa(t_i)}(t_i-)|^{1+p} \psi(\pi(t_i))$$

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Theorem 2.1 (K., 2012)

Assume that

$$\int_{\mathcal{P}} \mathbb{E}\left(\psi(\pi)\right) \mu(d\pi) < \infty$$

and let $p < p^*$. *Then*

$$\eta^{p^*-p} \mathcal{E}_p(\eta) \to \frac{\Lambda_0(p^*)}{\Phi'(p^*)(p^* - p)} \int_{\mathcal{P}} \mathbb{E}\left(\psi(\pi)\right) \mu(d\pi)$$

$\mathbb{P}$*–a.s. and in* $\mathscr{L}^1(\mathbb{P})$ *as* $\eta \downarrow 0$.

Theorem 2.2 (K., 2012)

Let $f:(0,1]\to\mathbb{R}_0^+$ *be a bounded and measurable function. Then*

$$\sum_{k\in\mathbb{N}}\lambda_{\eta,k}^{1+p^*}f\left(\frac{\lambda_{\eta,k}}{\eta}\right)\to\frac{\Lambda_0(p^*)}{\Phi'(p^*)}\int_{(0,1)}f(u)\int_{\mathcal{S}}\sum_{k\in\mathbb{N}}\mathbb{1}_{\{s_k<u\}}s_k^{1+p^*}\nu(ds)\frac{du}{u}$$

holds $\mathbb{P}$*-a.s. and in* $\mathscr{L}^1(\mathbb{P})$ *as* $\eta\downarrow 0$.

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holds $\mathbb{P}$-a.s. and in $\mathscr{L}^1(\mathbb{P})$ as $\eta\downarrow 0$.

Note that

$$\sum_{k\in\mathbb{N}}\lambda_{\eta,k}^{1+p^*}f\left(\frac{\lambda_{\eta,k}}{\eta}\right)=\int_{(0,1)}f\,\mathrm{d}\rho_\eta,$$

where

$$\rho_\eta:=\sum_{k\in\mathbb{N}}\lambda_{\eta,k}^{1+p^*}\delta_{\frac{\lambda_{\eta,k}}{\eta}}.$$

For every $\pi \in \mathcal{P}$ let $(\phi(x,\pi))_{x \in \mathbb{R}_0^+}$ be an $\mathbb{R}_0^+$–valued stochastic process, with

$$\forall\, x > 1 : \quad \phi(x,\pi) = 0,$$

which has càdlàg paths and is independent of Π.

We refer to $\phi := (\phi(x,\pi))_{x \in \mathbb{R}_0^+, \pi \in \mathcal{P}}$ as *random characteristic*.

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For every $\eta \in (0,1]$ set

$$Z_\eta^\phi := \sum_{i\in\mathcal{I}} \phi^{(i)}\left(\frac{\eta}{|\Pi_{\kappa(t_i)}(t_i-)|}, \pi(t_i)\right) \mathbb{1}_{\{|\Pi_{\kappa(t_i)}(t_i-)|>0\}}.$$

where the $\phi^{(i)}$ are independent copies of ϕ.

The process $(Z_\eta^\phi)_{\eta\in(0,1]}$ is said to be *counted with the characteristic* ϕ.

Theorem 2.3 (K., 2012)

Let ϕ be a random characteristic such that

$$\int_{\mathcal{P}} \mathbb{E}\left(\sup_{\eta\in(0,1]} \eta^{1+\tilde{p}}\phi\left(\eta,\pi\right)\right)\mu(d\pi) < \infty \qquad (1)$$

holds for some $\tilde{p} \in (\underline{p}, p^*)$. Then

$$\lim_{\eta\downarrow 0}\left(\eta^{1+p^*} Z_\eta^\phi\right) = \frac{\Lambda_0(p^*)}{\Phi'(p^*)}\int_{\mathcal{P}}\int_{(0,1]} x^{p^*}\mathbb{E}\left(\phi\left(x,\pi\right)\right) dx\mu(d\pi)$$

$\mathbb{P}$–a.s. and in $\mathscr{L}^1(\mathbb{P})$.

The LHS in the statement of Theorem 2.1 corresponds to the random characteristic ϕ given by

$$\phi(x,\pi) = \mathbb{1}_{\{x\in(0,1]\}} x^{-(1+p)} \psi(\pi)$$

for every $x \in \mathbb{R}_0^+$ and $\pi \in \mathcal{P}$. I.e.,

$$\eta^{1+p^*} Z_\eta^\phi = \eta^{p^*-p} \mathcal{E}_p(\eta).$$

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$$\eta^{1+p^*} Z_\eta^\phi = \eta^{p^*-p} \mathcal{E}_p(\eta).$$

The LHS in the statement of Theorem 2.2 corresponds to the random characteristic ϕ given by

$$\phi(x,\pi) = \sum_{n\in\mathbb{N}} \mathbb{1}_{\{|\pi_n|<x\le 1\}} x^{-(1+p^*)} |\pi_n|^{1+p^*} f\left(\frac{|\pi_n|}{x}\right)$$

for every $x \in \mathbb{R}_0^+$ and $\pi \in \mathcal{P}$. I.e.,

$$\eta^{1+p^*} Z_\eta^\phi = \sum_{k\in\mathbb{N}} \lambda_{\eta,k}^{1+p^*} f\left(\frac{\lambda_{\eta,k}}{\eta}\right).$$

Proof of Theorem 2.1

The proof is based on Theorem 2.3 and the random characteristic ϕ given by

$$\phi(x,\pi) = \mathbb{1}_{\{x\in(0,1]\}} x^{-(1+p)} \psi(\pi) \tag{2}$$

for all $x \in \mathbb{R}_0^+$ and $\pi \in \mathcal{P}$.

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Observe that for every $\eta \in (0,1]$ we have

$$\begin{aligned}
\eta^{1+p^*} Z_\eta^\phi &= \eta^{1+p^*} \sum_{i\in\mathcal{I}} \phi^{(i)}\left(\frac{\eta}{|\Pi_{\kappa(t_i)}(t_i-)|}, \pi(t_i)\right) \\
&= \eta^{p^*-p} \sum_{i\in\mathcal{I}} \mathbb{1}_{\{|\Pi_{\kappa(t_i)}(t_i-)|\geq\eta\}} |\Pi_{\kappa(t_i)}(t_i-)|^{1+p} \psi^{(i)}(\pi(t_i)) \\
&= \eta^{p^*-p} \mathcal{E}_p(\eta),
\end{aligned}$$

where the $\phi^{(i)}$ and $\psi^{(i)}$ are independent copies of ϕ and ψ respectively.

Proof of Theorem 2.1

Recall that

$$\phi(\eta,\pi) = \mathbb{1}_{\{\eta\in(0,1]\}}\eta^{-(1+p)}\psi(\pi).$$

Note that

$$\int_{\mathcal{P}} \mathbb{E}\left(\sup_{\eta\in(0,1]} \eta^{1+\tilde{p}}\phi(\eta,\pi)\right)\mu(\mathrm{d}\pi) = \int_{\mathcal{P}} \mathbb{E}\left(\sup_{\eta\in(0,1]} \eta^{\tilde{p}-p}\psi(\pi)\right)\mu(\mathrm{d}\pi) \leq \int_{\mathcal{P}} \mathbb{E}(\psi(\pi))\mu(\mathrm{d}\pi) < \infty$$

for every $\tilde{p} \in (p, p^*)$, i.e. the characteristic ϕ given by (2) satisfies (1).

Hence, the statement of Theorem 2.1 follows from Theorem 2.3. □

Proof of Theorem 2.2

For $p \in (\underline{p}, p^*)$ consider the function $f : (0,1) \to (1,\infty)$ given by

$$\forall x \in (0,1) : \quad f(x) = x^{p-p^*}.$$

Then we have

$$\sum_{k\in\mathbb{N}} \lambda_{\eta,k}^{1+p^*} f\left(\frac{\lambda_{\eta,k}}{\eta}\right)$$

$$= \eta^{p^*-p} \sum_{k\in\mathbb{N}} \lambda_{\eta,k}^{1+p}$$

$$= \eta^{p^*-p} \sum_{i\in\mathcal{I}} \mathbb{1}_{\{|\Pi_{\kappa(\mathbf{t}_i)}(t_i-)|\geq\eta\}} |\Pi_{\kappa(t_i)}(t_i-)|^{1+p} \sum_{j\in\mathbb{N}} |\pi_j(t_i)|^{1+p} \mathbb{1}_{\{|\Pi_{\kappa(\mathbf{t}_i)}(t_i-)\cap\pi_j(t_i)|<\eta\}}$$

$$= \eta^{p^*-p} \left(\sum_{i\in\mathcal{I}} \mathbb{1}_{\{|\Pi_{\kappa(\mathbf{t}_i)}(t_i-)|\geq\eta\}} |\Pi_{\kappa(t_i)}(t_i-)|^{1+p} \left(\sum_{j\in\mathbb{N}} |\pi_j(t_i)|^{1+p} - 1 \right) + 1 \right)$$

$$= \eta^{p^*-p} \left(\mathcal{E}_p(\eta) + 1\right).$$

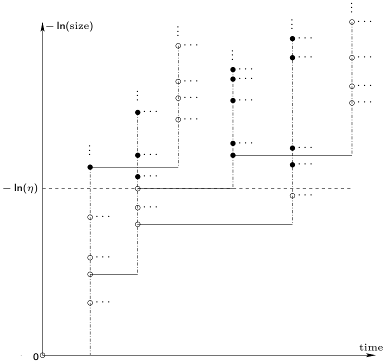
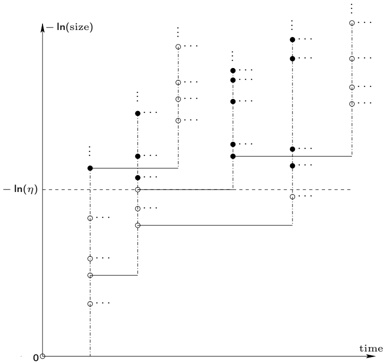


Figure: Illustration of Π and the stopping line at η.



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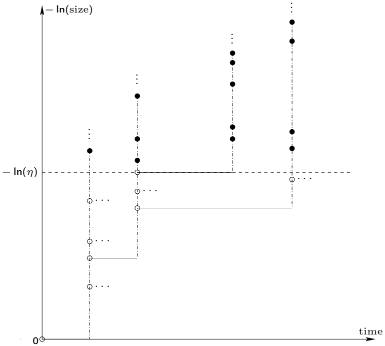

Figure: Illustration of the stopped process $(\lambda_{\eta,k})_{k\in\mathbb{N}}$.