Reciprocal classes of Jump processes

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Reciprocal probabilities

$\tilde{\mathbb{P}}$ is a probability measure on $\mathbb{D}([0,1] \times \mathcal{X})$, $\mathcal{X}$ Polish

Definition

$\tilde{\mathbb{P}}$ is said to be **reciprocal** iff for all $0 < s < t < 1$

$$\tilde{\mathbb{P}}\left(A \,|\sigma(X_u, u \in [0,s] \cup [t,1]) \right) = \tilde{\mathbb{P}}(A|\omega_s, \omega_t)$$

- Markov probability measures are reciprocal but the converse is false.
- Reciprocal probability measures were introduced by Bernstein (1932). Jamison began their rigorous mathematical treatment in a series of papers (1970,1974,1975).

Reciprocal family of a Markov process

Let us fix reference Markov probability measure $\mathbb{P}$ on $\mathbb{D}([0,1] \times \mathcal{X})$.

Definition (Reciprocal family)

$$\Re(\mathbb{P}) = \left\{ \tilde{\mathbb{P}} : \tilde{\mathbb{P}} << \mathbb{P} \text{ and } \exists h \text{ measurable }, \frac{d\tilde{\mathbb{P}}}{d\mathbb{P}} = h(X_0, X_1) \right\}$$

$\Re(\mathbb{P})$ is the set of probability measures dominated by $\mathbb{P}$ whose bridges are bridges of $\mathbb{P}$. Indeed,

$$\tilde{\mathbb{P}} \in \Re(\mathbb{P}) \Leftrightarrow \tilde{\mathbb{P}}(.) = \int_{\mathcal{X} \times \mathcal{X}} \mathbb{P}^{x,y}(.)\, h(x,y)\, \mathbb{P}(X_0 \in dx, X_1 \in dy)$$

- The elements of the reciprocal family are reciprocal probability measures.
- The notion of reciprocal class is closely related to the $h-$transform of a Markov process, as introduced by Doob (1953)

When do two Compound Poisson Process have the same bridges ?

Let us fix a finite (jump) set $\mathbf{E} = \{\mathbf{e^1}, ..., \mathbf{e^n}\} \subseteq \mathbb{R}^d$.

Definition

$c = (c_1, \cdots, c_n) \in \mathbb{Z}^n$ is a **cycle** of $\mathbf{E}$ iff

$$\sum_{j=1}^{n} c_j \mathbf{e^j} = 0$$

We denote the set of all cycles of $\mathbf{E}$ by $\mathcal{C}$.

The definition of a cycle depends on the geometrical properties of the jump set $\mathbf{E}$.

- $\mathbb{P}$ is the CPP with Lévy measure $\nu := \sum_{\mathbf{j=1}}^{\mathbf{n}} \nu_{\mathbf{j}} \delta_{\mathbf{e_j}}, \nu_j > 0$, started at 0.
- $\tilde{\mathbb{P}}$ is the CPP with Lévy measure $\tilde{\nu} := \sum_{\mathbf{j=1}}^{\mathbf{n}} \tilde{\nu}_{\mathbf{j}} \delta_{\mathbf{e_j}}$, started at 0

Proposition

The following properties are equivalent:

- $\tilde{\mathbb{P}} \in \Re(\mathbb{P})$
- The "cycle invariants" of $\mathbb{P}$ and $\tilde{\mathbb{P}}$ coincide:

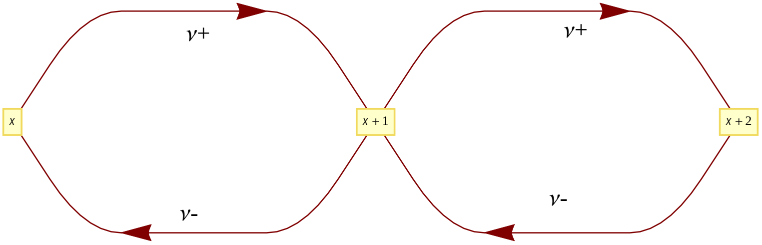
$$\forall \; cycle \; cin\mathcal{C}, \quad \prod_{j=1}^{n} \nu_j^{c_j} = \prod_{j=1}^{n} \tilde{\nu}_j^{c_j}$$

- $g \in \mathcal{C}^{\perp}$, where $g_j := \log \frac{\tilde{\nu}_j}{\nu_j}, j = 1, \cdots, n$:

Examples

Let $\mathbf{E} = \{-1, 1\}$. Then, with $c = (1, 1)$,

$$\tilde{\mathbb{P}} \in \Re(\mathbb{P}) \Leftrightarrow \nu_{-1}\nu_1 = \tilde{\nu}_{-1}\tilde{\nu}_1$$



Let $\mathbf{E} = \left\{-1, \sqrt{2}\right\}$. No cycles in $\mathbf{E}$, therefore

$$\tilde{\mathbb{P}} \in \Re(\mathbb{P}), \quad \forall\ \tilde{\nu}_{-1}, \tilde{\nu}_{\sqrt{2}}$$

and the reciprocal family of $\mathbb{P}$ contains all CPP like $\tilde{\mathbb{P}}$.

Example: multidimensional case

Let $\mathbf{E} = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \end{pmatrix} \right\} \subseteq \mathbb{R}^2$. Then:

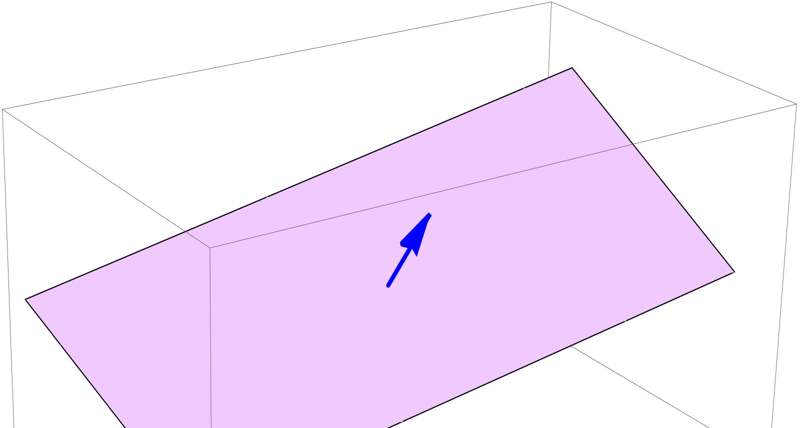
$$\tilde{\mathbb{P}} \in \Re(\mathbb{P}) \Leftrightarrow \nu_1 \nu_2 \nu_3 = \tilde{\nu}_1 \tilde{\nu}_2 \tilde{\nu}_3$$

The geometric approach: an example

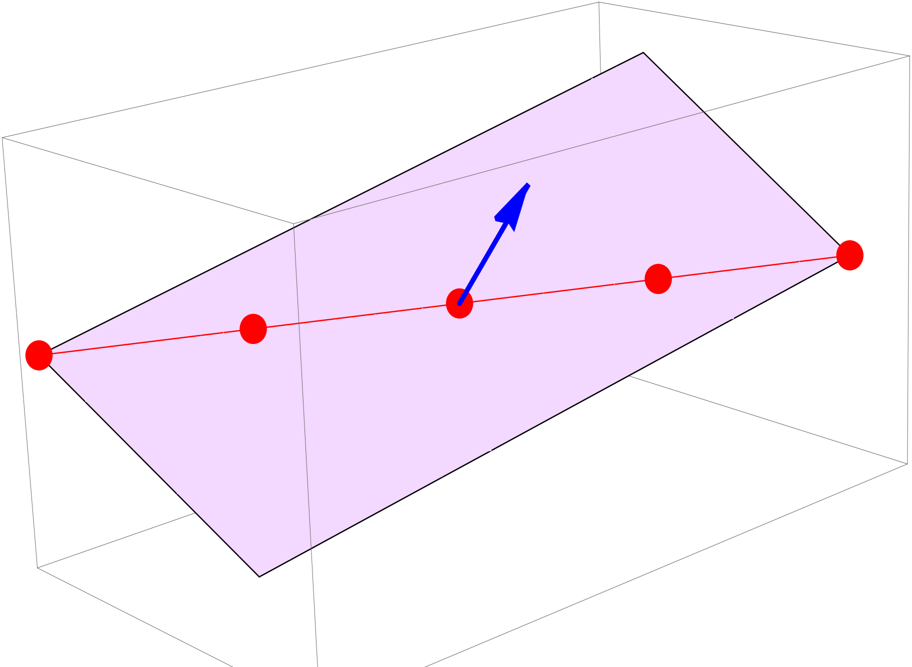
Let $\mathbf{E} = \{1, 2, \sqrt{2}\}$ and $\nu := \nu_1\delta_1 + \nu_2\delta_2 + \nu_3\delta_{\sqrt{2}}$.

Question: How to find all Lévy measures $\tilde{\nu}$ such that $\tilde{\mathbb{P}} \in \Re(\mathbb{P})$?

1) Represent **E** as vector of $\mathbb{R}^3$ and consider its orthogonal complement $\mathbf{E}^\perp$:

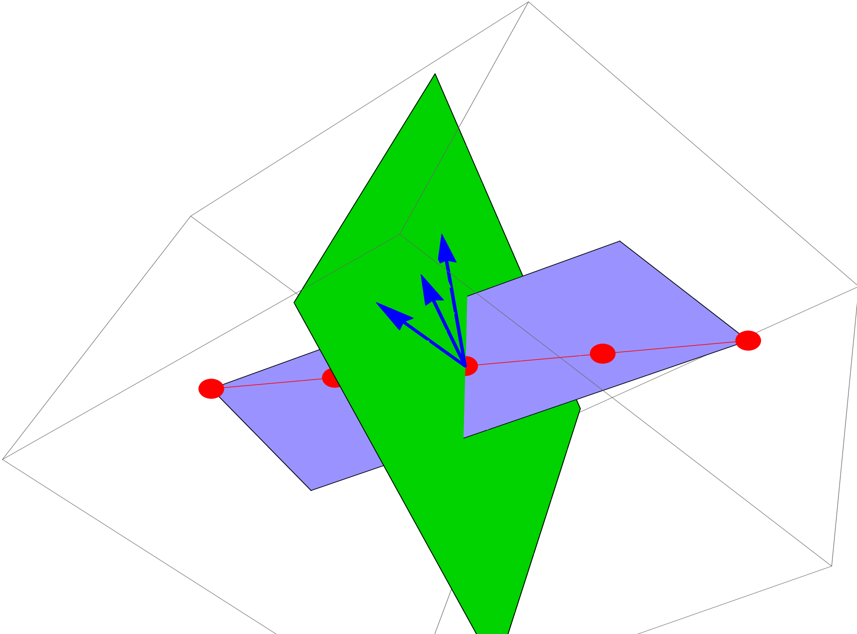


2) Consider $\mathcal{C} := E^{\perp} \cap \mathbb{Z}^3$.



3) Dualize a second time to obtain $\mathcal{C}^{\perp} = \left(\mathbf{E}^{\perp} \cap \mathbb{Z}^{3}\right)^{\perp}$

4) Pick any $v \in \mathcal{C}^{\perp}$ Define $\tilde{\nu}_j := e^{v_j} \nu_j, \quad j = 1, 2, 3.$ Then $\tilde{\mathbb{P}} \in \Re(\mathbb{P})$.



Few consequences of the geometric characterization

Let $\Phi : \mathbb{R}^d \to \mathbb{R}^m$ be a linear map and consider the image ${}^{\Phi}\nu$ of the Lévy measure ν under Φ:

$$ {}^{\Phi}\nu(e) = \nu(\Phi^{-1}(e)) $$

The corresponding CPP on $\mathbb{R}^m$ is denoted by ${}^{\Phi}\mathbb{P}$

Proposition

i) Let Φ be **angle-preserving (conformal)** :

$$ \tilde{\mathbb{P}} \in \Re(\mathbb{P}) \quad \Leftrightarrow \quad {}^{\Phi}\tilde{\mathbb{P}} \in \Re({}^{\Phi}\mathbb{P}) $$

ii) Let Φ be a **projection** :

$$ \tilde{\mathbb{P}} \in \Re(\mathbb{P}) \ \Rightarrow \ {}^{\Phi}\tilde{\mathbb{P}} \in \Re({}^{\Phi}\mathbb{P}) $$

Comparison of reciprocal families of two general CPP

We consider two CPP $\mathbb{P}$ and $\tilde{\mathbb{P}}$ with general jump measures resp. $\nu, \tilde{\nu}$ and initial distribution δ_0.

Theorem

$\tilde{\mathbb{P}} \in \Re(\mathbb{P})$ iff

i) $\tilde{\nu} << \nu$

ii) There exists a measurable function Ψ such that:

$$\forall m \in \mathbb{N} \quad \prod_{j=1}^{m} G(x^j) = \Psi\left(\sum_{j=1}^{m} x^j\right) \ \nu^{\otimes_n} - a.e.$$

where $G := \frac{d\tilde{\nu}}{d\nu}$.

Towards an implicit description of the reciprocal family

- The search for reciprocal invariants is a key step in the study of reciprocal families. In the framework of processes with continuous paths, and more precisely Brownian diffusions, Clark (1990) was able to exhibit two invariant functionals.
- It is important to find also a characterization of the non-Markov probability measures of a reciprocal family, possibly as solution of a functional equation.
- Such type of equation, should contain in its formulation the reciprocal invariants of the reciprocal family we are looking for, and possibly provide an interpretation for them.

Towards an implicit description of the reciprocal family

- In the framework of Brownian diffusions Roelly and Thieullen (2002,2005) used integration by parts formulae that put in duality the Malliavin derivative operator with the stochastic integral. Under a loop condition on the test functions such formulae characterize reciprocal families.
- Murr (2012) studied the case of counting processes and characterizes their reciprocal classes using stochastic derivatives associated to perturbation of the jump times.
- Process with jumps of different sizes seems to require the introduction of new types of transformations (work in progress)

Thank you for attention