Gaussian estimates for Schrödinger perturbations

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Motivation

Let $x, y \in \mathbb{R}^d$. For $s < t$,

$$g(s, x, t, y) = [4\pi(t - s)]^{-d/2} \exp\left(-|y - x|^2/(t - s)\right),$$

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$$\int_{\mathbb{R}^d} g(s, x, u, z) g(u, z, t, y) dz = g(s, x, t, y), \quad \text{if} \quad s < u < t.$$

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Fundamental solution:

$$\int_{\mathbb{R}} \int_{\mathbb{R}^d} g(s,x,u,z)\left[\partial_u + \Delta_z\right]\phi(u,z)dzdu = -\phi(s,x),$$

for $s \in \mathbb{R}$, $x \in \mathbb{R}^d$ and $\phi \in C_c^\infty(\mathbb{R} \times \mathbb{R}^d)$.

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GOAL: Given function $q \geq 0$ on *time-space*, find a transition density $\tilde{g}$ such that for all $s \in \mathbb{R}$, $x \in \mathbb{R}^d$, $\phi \in C_c^\infty(\mathbb{R} \times \mathbb{R}^d)$,

$$\int_{\mathbb{R}} \int_{\mathbb{R}^d} \tilde{g}(s,x,u,z)\big[\partial_u + \Delta_z + q(u,z)\big]\phi(u,z)dzdu = -\phi(s,x).$$

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$$\tilde{g}(s, x, t, y) = g(s, x, t, y) + \int_s^t \int_{\mathbb{R}^d} g(s, x, u, z) q(u, z) \tilde{g}(u, z, t, y) dz du,$$

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Here $\quad g_n = (gq)^n g = (gq)g_{n-1} \quad$ and $\quad g_0 = g$.

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Definition

Function $p \colon \mathbb{R} \times X \times \mathbb{R} \times X \to [0, \infty]$ *is called a transition density if*

$$\int_X p(s,x,u,z)p(u,z,t,y)\, dz = p(s,x,t,y)\,, \quad s < u < t.$$

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Definition (Schrödinger perturbation of p by q)

$$\tilde{p}(s,x,t,y) = \sum_{n=0}^{\infty} p_n(s,x,t,y)\,.$$

Local smallness, global growth control

Assume that for all $s < t$, $x, y \in X$,

$$p_1(s,x,t,y) \le \big[\eta + Q(s,t)\big]\, p(s,x,t,y), \qquad (\bigstar)$$

where $\eta \ge 0$ and $0 \le Q(s,u) + Q(u,t) \le Q(s,t)$, if $s < u < t$.

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Theorem (T. Jakubowski, 2009)

If $(\bigstar)$ holds, then for all $s < t$, $x, y \in X$,

$$\tilde{p}(s,x,t,y) \leq \left(\frac{1}{1-\eta}\right)^{1+\frac{Q(s,t)}{\eta}} p(s,x,t,y)\,, \qquad \text{if} \quad 0 < \eta < 1\,,$$

and

$$\tilde{p}(s,x,t,y) \leq e^{Q(s,t)} p(s,x,t,y)\,, \qquad \text{if} \quad \eta = 0\,.$$

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We then verify $(\bigstar)$ by considering

$$\frac{p_1(s,x,t,y)}{p(s,x,t,y)} \leq c \int_s^t \int_X \Big[p(s,x,u,z) + p(u,z,t,y)\Big] q(u,z)dzdu.$$

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3G theorems

For instance, let p be α-stable transition density, $\alpha \in (0,2)$. Then

$$p(s,x,t,y) \approx (t-s)^{-d/\alpha} \wedge \frac{t-s}{|y-x|^{d+\alpha}}\,.$$

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3G does not hold for g:

$z-x=y-z=\xi$, $u-s=t-u=\tau$

$$g(s,x,u,z) \wedge g(u,z,t,y) = (4\pi\tau)^{-d/2} \exp\left(-|\xi|^2/4\tau\right),$$

$$g(s,x,t,y) = (8\pi\tau)^{-d/2} \exp\left(-|\xi|^2/2\tau\right).$$

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Q. S. Zhang- 2003

A sharp comparison result concerning schrödinger heat kernels.

A majorant

Consider auxiliary transition density p^* and $C \geq 1$ such that

$$p(s,x,t,y) \leq C\, p^*(s,x,t,y)\,, \qquad s < t,\ x,y \in X\,. \qquad (\bigstar\bigstar)$$

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Definition

We say that $q \in \mathcal{N}(p,p^*,C,\eta,Q)$, *if* $(\bigstar\bigstar)$ *holds and*

$$\int_s^t \int_X p(s,z,u,z) q(u,z) p^*(u,z,t,y)\, dzdu \qquad (\bigstar\bigstar\bigstar)$$
$$\leq \big[\eta + Q(s,t)\big]\, p^*(s,x,t,y),$$

where $\eta \geq 0$, Q *is continuous and superadditive.*

Q is called *superadditive* if

$$0 \leq Q(s,u) + Q(u,t) \leq Q(s,t)\,, \qquad s<u<t$$

The main result

Theorem

If $q \in \mathcal{N}(p, p^*, C, \eta, Q)$ *and* $\eta \in [0,1)$, *then for all* $s < t$, $x, y \in X$ *and* $\varepsilon \in (0, 1-\eta)$,

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Remark. If we let $\varepsilon = (1-\eta)/2$,

$$\left(\frac{2C}{1-\eta} \right)^{1+\frac{2Q(s,t)}{1-\eta}} .$$

If $0 < \eta < 1/2$, we may take $\varepsilon = \eta$,

$$\left(\frac{C}{1-2\eta} \right)^{1+\frac{Q(s,t)}{\eta}} .$$

Gaussian transition density

For $c > 0$ consider

$$g_c(s, x, t, y) = [4\pi(t - s)/c]^{-d/2} \exp\left(-c|y - x|^2/(t - s)\right).$$

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Aim:
Propose convenient conditions sufficient for $q \in \mathcal{N}(g_b, g_a, C, \eta, Q)$.

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Theorem (sharp 4G)

Let $M = [b/(b-a)]^{d/2} \exp\left[\frac{d}{2} L(\frac{a}{b-a})\right]$, *then*

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Notice. $M = (1 - a/b)^{-d}$ provided $1/(1+e^{-1/2}) \leq a/b < 1$.

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We verify (★★★) by

$$\int_s^t \int_{\mathbb{R}^d} g_b(s,u,z) q(u,z) g_a(u,z,t,y) dz du$$
$$\leq g_a(s,x,t,y)\, M \int_s^t \int_{\mathbb{R}^d} \left[g_{b-a}(s,x,u,z) + g_a(u,z,t,y) \right] q(u,z) dz du.$$

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Examples

For $c > 0$, $h > 0$ and $V : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ we denote

$$N_h^c(V) = \sup_{s,x} \int_s^{s+h} \int_{\mathbb{R}^d} g_c(s,x,u,z)|V(u,z)|dzdu + \sup_{t,y} \int_{t-h}^{t} \int_{\mathbb{R}^d} g_c(u,z,t,y)|V(u,z)|dzdu.$$

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$$\eta = c' M\, N_h^c(q)\,, \qquad Q(s,t) = \eta(t-s)/h\,.$$

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- parabolic Kato class, $\lim_{h \to 0^+} N_h^c(q) = 0$,
- Kato class $\mathcal{K}_d$, $q(u,z) = q(z)$,
- $q(u,z) = q(z)$ that is not in (parabolic) Kato class,
- $q(u) = u \mathbb{1}_{u \geq 0}$, $N_h^c(q) = \infty$, $\eta = 0$, $Q(s,t) = M(t^2 - s^2)/2$.

For $U \colon \mathbb{R}^d \to \mathbb{R}$ we denote

$$I_\delta(U) = \sup_{x \in \mathbb{R}^d} \int_{|z-x|<\delta} \frac{|U(z)|}{|z-x|^{d-2}}\, dz\,, \quad \delta > 0\,,$$

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There are $C_0 = C_0(d,c)$ and $C_1 = C_1(d,c)$ such that for all $h > 0$ and U,

$$C_0\, I_{\sqrt{h}}(U) \leq \sup_{s \in \mathbb{R},\, x \in \mathbb{R}^d} \int_s^{s+h} \int_{\mathbb{R}^d} g_c(s,x,u,z)|U(z)|dzdu \leq C_1\, I_{\sqrt{h}}(U)\,.$$

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For $c_0 = c_0(d) = \frac{\Gamma(d/2-1)}{4\pi^{d/2}}$, $c > 0$, $\tau > 0$, $h > 0$ and U,

$$\sup_{s \in \mathbb{R}, x \in \mathbb{R}^d} \int_s^{s+\tau} \int_{\mathbb{R}^d} g_c(s,x,u,z)|U(z)|\, dzdu \leq \left(c\, c_0 + \frac{\tau}{h|B(0,1/2)|} \right) I_{\sqrt{h}}(U)\,.$$

Kato class, $d \geq 3$

For $U : \mathbb{R}^d \to \mathbb{R}$ we denote

$$I_\delta(U) = \sup_{x \in \mathbb{R}^d} \int_{|z-x|<\delta} \frac{|U(z)|}{|z-x|^{d-2}}\, dz\,, \quad \delta > 0\,,$$

Fact

$$q \in \mathcal{K}_d \quad \textit{if and olny if} \;\; \lim_{\delta \to 0^+} I_\delta(q) = 0$$

For $c_0 = c_0(d) = \frac{\Gamma(d/2-1)}{4\pi^{d/2}}$,

$$\eta = b\, c_0(d) M\, I_{\sqrt{h}}(q)\,,$$

$$Q(s,t) = (t-s) I_{\sqrt{h}}(q)\, 2M/(|B(0,1/2)|h)\,.$$

An application

Theorem

Let $d \geq 1$. Assume that $b > 0$, $\Lambda \geq 1$ and $\lambda \in \mathbb{R}$ exist such that for $s < t$,

$$p(s,x,t,y) \leq \Lambda e^{\lambda(t-s)} g_b(s,x,t,y), \quad x, y \in \mathbb{R}^d .$$

Let $0 < a < b$ and $C = \Lambda(b/a)^{d/2}$. If $q \in \mathcal{N}(g_b, g_a, \frac{C}{\Lambda}, \frac{\eta}{\Lambda}, \frac{Q}{\Lambda})$, then for all $s < t$, $x, y \in \mathbb{R}^d$,

$$\tilde{p}(s,x,t,y) \leq \left(\frac{C}{1 - \eta - \varepsilon} \right)^{1 + \frac{Q(s,t)}{\varepsilon}} e^{\lambda(t-s)} g_a(s,x,t,y).$$

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Let $\quad p(s,x,t,y) = g_1(s,x-s,t,y-t)\,, \quad s < t,\ x,y \in \mathbb{R}.$

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There are no constants c_1, c_2 such that

$$p(s,x,t,y) \leq c_1 g_{c_2}(s,x,t,y)\,,$$

for all $s < t$ and $x,y \in \mathbb{R}$.

An application

Theorem

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Let $\quad p(s,x,t,y) = g_1(s, x-s, t, y-t) \,, \quad s < t, \; x, y \in \mathbb{R}.$
For each $b \in (0,1)$ we have

$$p(s,x,t,y) \leq b^{-1/2} e^{\frac{b}{4(1-b)}(t-s)} g_b(s,x,t,y) \,,$$

for all $s < t$, $x, y \in \mathbb{R}$.

Elliptic operator

E. B. Dynkin, Diffusions, superdiffusions and partial differential equations, volume 50 of American Mathematical Society Colloquium Publications. American Mathematical Society, Providence, RI, 2002.

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Let $f : \mathbb{R} \times \mathbb{R}^d \ni (s, x) \mapsto \mathbb{R}$, and

$$Lf = \sum_{i,j=1}^{n} a_{ij}(s,x) \frac{\partial^2 f}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(s,x) \frac{\partial f}{\partial x_i} .$$

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Let $p(s, x, t, y)$ be the fundamental solution of

$$\frac{\partial f}{\partial s} + Lf = 0\,.$$

Elliptic operator

For all $s \in \mathbb{R}$, $x \in \mathbb{R}^d$ and $\phi \in C_c^\infty(\mathbb{R} \times \mathbb{R}^d)$, we have

$$\int_s^\infty \int_{\mathbb{R}^d} p(s,x,u,z) \left[\frac{\partial \phi(u,z)}{\partial u} + L\phi(u,z) \right] dzdu = -\phi(s,x) \,.$$

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The perturbed transition density $\tilde{p}$ corresponds to the operator $L + q$: for all $s \in \mathbb{R}$, $x \in \mathbb{R}^d$ and $\phi \in C_c^\infty(\mathbb{R} \times \mathbb{R}^d)$,

$$\int_s^\infty \int_{\mathbb{R}^d} \tilde{p}(s,x,u,z) \left[\frac{\partial \phi(u,z)}{\partial u} + L\phi(u,z) + q(u,z)\phi(u,z) \right] dzdu = -\phi(s,x) \,.$$

Thank you!