An Itō formula for convoluted Lévy processes

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- Let $(L(t))_{t\geq 0}$ be a **Lévy process** on the **white noise probability space** $(\mathcal{S}', \mathcal{B}, \Lambda)$.
  Here $\mathcal{S}'$ denotes the space of **tempered distributions** (the dual of the Schwartz space $\mathcal{S}$), $\mathcal{B}$ the Borel σ-algebra and Λ the **white noise measure** whose existence is guaranteed by the Bochner-Minlos theorem.
- The **characteristic triplet** of the Lévy process is (γ, σ^2, ν) with a drift $\gamma \in \mathbb{R}$, diffusion coefficient $\sigma^2 > 0$ (i.e. we have a Brownian part) and Lévy measure ν.
- We assume

$$\int_{|x|\geq 1} x^k \ \nu(dx) < \infty$$

  for all $k \in \mathbb{N}_{\geq 1}$, that is, **all moments of** $L(t)$ exist.
- Furthermore we suppose $\mathbb{E}[L(1)] = 0$, so $\gamma = -\int_{|x|>1} x \ \nu(dx)$.
- In our setting, L is a **martingale**.

- Define $\mathbb{R}_0 := \mathbb{R} \setminus \{0\}$ and let for measurable $B \subseteq \mathbb{R}_0 \times \mathbb{R}$

$$N(B) := \#\{s \geq 0 : \ (L(s) - L(s^-), s) \in B\}$$

  be the **Poisson random measure** with **intensity measure** $\nu(dx)\, ds$.
- From now on we consider a **two-sided Lévy process** $L = (L(t))_{t \in \mathbb{R}}$ that is constructed by taking two independent copies $(L_1(t))_{t \geq 0}$ and $(L_2(t))_{t \geq 0}$ of a one-sided Lévy process and setting

$$L(t) := \begin{cases} L_1(t), & t \geq 0, \\ -L_2(-t^-), & t < 0. \end{cases}$$

Convoluted Lévy processes

Definition

We call a stochastic process $M = (M(t))_{t \in \mathbb{R}}$ given by

$$M(t) = \int_{\mathbb{R}} f(t,s)\ L(ds), \quad t \in \mathbb{R},$$

a **convoluted Lévy process** with kernel f. The function $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ satisfies

$$f(t,\cdot) \in L^2(\mathbb{R}),\ t \in \mathbb{R}.$$

Definition

From now on we restrict ourselves to the class $\mathcal{K}$ of kernel funcions that fulfil

1. $f(t,\cdot) \in L^n(\mathbb{R}),\ n \geq 2,\ t \in \mathbb{R},$
2. $f(t,s) = 0$ for $s > t \geq 0$ (kernel of **Volterra** type),
3. $f(0,s) = 0$ for almost all s (which gives $M(0) = 0$).
4. plus some rather technical conditions on a set $[0,u) \times \mathbb{R}$ for some $u \in \mathbb{R}_{>0} \cup \{\infty\}$.

Lemma

For each $f \in \mathcal{K}$ we have $M(t) \in L^p(\mathcal{S}', \Lambda)$ for every $p \geq 1$.

Remark

Note that the class $\mathcal{K}$ contains kernels that lead to

- *one-sided* ***shot noise processes****,*
- *one-sided* ***Ornstein-Uhlenbeck type processes*** *and*
- ***fractional Lévy processes*** *with kernel*

$$f(t,s) = \frac{1}{\Gamma(d+1)}\left((t-s)_+^d - (-s)_+^d\right)$$

of ***Mandelbrot-Van Ness*** *type and* ***fractional integration parameter*** $d \in (0, \frac{1}{2})$ *that corresponds to a* ***Hurst parameter*** $H = d + \frac{1}{2} \in (\frac{1}{2}, 1)$.

The S-transform

- Define $g^*(x,t) := x \cdot g(x,t)$ and denote by Ξ the set

$$\Xi := \operatorname{span}\{g \in L^2(\mathbb{R}^2, x^2\nu(dx) \times \lambda(dt)) : \; g^* \in L^1(\nu \otimes \lambda), \\ g(x,t) = g_1(x) \cdot g_2(t) : g_1 \in L^\infty \text{ and } g_2 \in \mathcal{S}\}.$$

- Now introduce for $g \in \Xi$ the so called **Wick exponential** by

$$\mathcal{E}(g) := \sum_{n=0}^{\infty} \frac{I_n(g^{\otimes n})}{n!},$$

where I_n denotes the **multiple Lévy-Itō integral** of order n and $g^{\otimes n}$ the n-fold tensor product of g.
(See [Lee and Shih, 1999] for details and an explicit formula of $\mathcal{E}(g)$).

Definition

For $X \in L^2(\mathcal{S}', \Lambda)$, the S-**transform** SX is an integral transform defined on the set Ξ by

$$S(X)(g) := \mathbb{E}^{\mathcal{Q}_g}[X], \quad g \in \Xi,$$

with

$$d\mathcal{Q}_g := \mathcal{E}(g)\ d\Lambda.$$

Proposition (compare [Lee and Shih, 1999])

Let $X, Y \in L^2(\mathcal{S}', \Lambda)$ *with*

$$S(X)(g) = S(Y)(g)$$

for all $g \in \Xi$. *Then* $X = Y$ Λ*-a.s.*

Hitsuda-Skorokhod integrals

We start with a motivating lemma.

Lemma (compare [Bender, 2003] and [Bender and Marquardt, 2008])

Let $0 \leq a \leq b$ *and* $X : [a,b] \times \mathcal{S}' \to \mathbb{R}$ *be a predictable process such that*

$$\mathbb{E}\left[\int_a^b |X(t)|^2 \, dt\right] < \infty.$$

Then the stochastic integral $\int_a^b X(s)\, L(ds)$ *is the unique square-integrable random variable in* $L^2(\mathcal{S}', \Lambda)$ *with* S*-transform given by*

$$\int_a^b S(X(t))(g) \frac{d}{dt} S(L(t))(g) \, dt, \quad g \in \Xi.$$

Definition

Suppose the mapping $t \mapsto S(M(t))(g)$ is differentiable for every $g \in \Xi$, $B \subseteq \mathbb{R}$ is a Borel set and $X : B \times \mathcal{S}' \to \mathbb{R}$ is a stochastic process such that $X(t)$ is square-integrable for each $t \in B$. The process X is said to have a **Hitsuda-Skorokhod integral with respect to M** if

$$t \mapsto S(X(t))(g) \frac{d}{dt} S(M(t))(g) \in L^1(B)$$

for all $g \in \Xi$ and there is a $\Phi \in L^2(\mathcal{S}', \Lambda)$ such that for all $g \in \Xi$

$$S(\Phi)(g) = \int_B S(X(t))(g) \frac{d}{dt} S(M(t))(g) \, dt.$$

In this case, Φ is uniquely determined by the injectivity of the S-transform and we write

$$\Phi = \int_B X(t) \; M^{\diamond}(dt).$$

Motivated by a similar result concerning the integral arising from the jump measure we define the other Hitsuda-Skorokhod integral that is needed to formulate our result.

Definition

Let $X : \mathbb{R}_0 \times [0, T] \times \mathcal{S}' \to \mathbb{R}$ be a random field. The **Hitsuda-Skorokhod integral** of X **with respect to the jump measure N** is said to exist in $L^2(\mathcal{S}', \Lambda)$ if there is a random variable $\Phi \in L^2(\mathcal{S}', \Lambda)$ that satisfies

$$S(\Phi)(g) = \int_0^T \int_{\mathbb{R}_0} S(X(y,t))(g)(1 + g^*(y,t)) \; \nu(dy) \; dt$$

for all $g \in \Xi$. We write

$$\Phi = \int_0^T \int_{\mathbb{R}_0} X(y,t) \; N^\diamond(dy, dt).$$

An Itō formula

Theorem (An Itō formula, Bender, Knobloch, O., 2013)

Let $T \in [0, u] \cap \mathbb{R}$ and $G \in C^2(\mathbb{R})$ with G, G' and G'' of polynomial growth. The formula

$$G(M(T)) = G(0) + \int_0^T G'(M(t^-))\ M^\diamond(dt) + \sigma^2 \int_0^T G''(M(t^-)) \left(\frac{f(t,t)^2}{2} + \int_{-\infty}^t f(t,s) \frac{d}{dt} f(t,s) ds \right) dt + \sum_{0<t\leq T} G(M(t)) - G(M(t^-)) - G'(M(t^-)) \Delta M(t) + \int_0^T \int_{-\infty}^t \int_{\mathbb{R}_0} (G'(M(t^-) + xf(t,s)) - G'(M(t^-))) \cdot x \frac{d}{dt} f(t,s)\ N^\diamond(dx, ds)\ dt$$

holds, provided all terms on the right hand side exist in $L^2(\mathcal{S}', \Lambda)$.

- To find the right hand side of the Itō formula explicitly we write

$$S(G(M(T)))(g) = G(0) + \int_0^T \frac{d}{dt} S(G(M(t)))(g)\, dt$$

and calculate $\frac{d}{dt} S(G(M(t)))(g)$.

- For that purpose we apply the inverse Fourier theorem in the special case $G \in \mathcal{S}$ (general case via approximation) to see

$$S(G(M(t)))(g) = \mathbb{E}^{\mathcal{Q}_g}[G(M(t))] = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \mathcal{F}G(u) \mathbb{E}^{\mathcal{Q}_g}[e^{iuM(t)}]\, du.$$

- Now we calculate $\frac{d}{dt}\mathbb{E}^{\mathcal{Q}_g}[e^{iuM(t)}]$, make use of some standard manipulations of the Fourier transform and integrate t from 0 to T.
- Finally we identify the resulting terms with the S-transforms of the summands on the right-hand side of the Itō formula and use injectivity.

For the calculation of $\frac{d}{dt}\mathbb{E}^{\mathcal{Q}_g}[e^{iuM(t)}]$ we apply the following result

Proposition

$$\mathbb{E}^{\mathcal{Q}_g}[e^{iuM(t)}] = \exp\Big\{ -\frac{\sigma^2 u^2}{2}\int_{-\infty}^{t} f(t,s)^2\, ds + iuS(M(t))(g) + \int_{-\infty}^{t}\int_{\mathbb{R}_0} [(e^{iuxf(t,s)} - 1 - iuxf(t,s))(1+g(x,s))]\,\nu(dx)\,ds\Big\}$$

and the mapping $u \mapsto \mathbb{E}^{\mathcal{Q}_g}[e^{iuM(t)}]$ *is a Schwartz function for* $t \neq 0$.

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