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An Evolution Operator for the Nonstationary Sobolev Type Equation

Consider the nonstationary equation

$$L\dot{u}(t) = M_t u(t), \qquad t \in \mathfrak{J} \subset \mathbb{R} \tag{1}$$

where operators $L \in \mathcal{L}(\mathfrak{U};\mathfrak{F})$, $M_t \in \mathcal{L}(\mathfrak{U};\mathfrak{F})$ for every $t \in \mathfrak{J}$. If $\ker L \neq \{0\}$ then (1) is called *Sobolev type equation* [1].

Definition 1. Sets $\rho^L(M_t) = \{\mu \in \mathbb{C} : (\mu L - M_t)^{-1} \in \mathcal{L}(\mathfrak{F};\mathfrak{U})\}$ and $\sigma^L(M_t) = \mathbb{C} \setminus \rho^L(M_t)$ are called *L-resolvent set* and *L-spectrum* of operator-function M_t correspondingly.

The operator-function M_t is called *spectrally bounded with respect to operator* L (or simply (L,σ)*-bounded*), if

$$\exists a_t \in C(\mathfrak{J};\mathbb{R}_+) \quad \forall t \in \mathfrak{J} \quad \max\{|\mu| : \ \mu \in \sigma^L(M_t)\} \leq a_t < +\infty.$$

Let the operator-function M_t be (L,σ)-bounded and the contour $\gamma_t=\{\mu\in\mathbb{C}:|\mu|=2a_t\}$. Consider integrals

$$P_t = \frac{1}{2\pi i}\int\limits_{\gamma_t} R^L_\mu(M_t)d\mu, \qquad Q_t = \frac{1}{2\pi i}\int\limits_{\gamma_t} L^L_\mu(M_t)d\mu.$$

Operators $P_t : \mathfrak{U} \to \mathfrak{U}$ and $Q_t : \mathfrak{F} \to \mathfrak{F}$ are projectors. It was proved in [1] with fixed $t \in \mathfrak{J}$.

Theorem 1. [2] *Let the operator-function $M_t \in \mathcal{L}(\mathfrak{U};\mathfrak{F})$ be (L,σ)-bounded. Then*

(i) *the action of operators $L_{t,k} : \mathfrak{U}^k_t \to \mathfrak{F}^k_t$, $M_{t,k} : \mathfrak{U}^k_t \to \mathfrak{F}^k_t$ $\forall t \in \mathfrak{J}$, $k = 0,1$ is observed*;

(ii) *there exists an operator $M^{-1}_{t,0} \in \mathcal{L}(\mathfrak{F}^0_t;\mathfrak{U}^0_t)$, $t \in \mathfrak{J}$, besides if the operator-function $M_t : \mathfrak{J} \to \mathcal{L}(\mathfrak{U};\mathfrak{F})$ is strongly differential then the operator-function $M^{-1}_{t,0}(I-Q_t) \in \mathcal{L}(\mathfrak{F};\mathfrak{U}^0_t)$ is also strongly differential by $t \in \mathfrak{J}$ and if the operator-function $\frac{d}{dt}M_t$ is strongly continuous then the operator-function $\frac{d}{dt}(M^{-1}_{t,0}(I-Q_t))$ is also strongly continuous by $t \in \mathfrak{J}$*;

(iii) *there exists an operator $L^{-1}_{t,1} \in \mathcal{L}(\mathfrak{F}^1_t;\mathfrak{U}^1_t)$, $t \in \mathfrak{J}$ where the operator-function $L^{-1}_{t,1}Q_t \in C(\mathfrak{J};\mathcal{L}(\mathfrak{F};\mathfrak{U}^1_t)$.*

Definition 2. The (L,σ)-bounded operator-function M_t is called $(L,0)$*-bounded* if $\forall t \in \mathfrak{J}$ $M^{-1}_{t,0}L_{t,0} = H_t \equiv \mathbb{O}$.

Theorem 2. [2] *Let the operator-function $M_t \in \mathcal{L}(\mathfrak{U};\mathfrak{F})$ be $(L,0)$-bounded. Then* $\ker L = \mathfrak{U}^0_t$, $\operatorname{im} L = \mathfrak{F}^1_t$ *for all* $t \in \mathfrak{J}$.

Set $\ker L = \ker P_t = \mathfrak{U}^0$, $\ker Q_t = \mathfrak{F}^0_t$; $\operatorname{im} P_t = \mathfrak{U}^1_t$ and $\operatorname{im} L = \operatorname{im} Q_t = \mathfrak{F}^1$. By L_0 $(M_{t,0})$ denote the restriction of operator L (M_t) on $\mathfrak{U}^0$ and by $L_{t,1}$ $(M_{t,1})$ the restriction of operator L (M_t) on $\mathfrak{U}^1_t$, $t \in \mathfrak{J}$.

The vector-function $u\in C^1(\mathfrak{J};\mathfrak{U})$ satisfying (1) is called *the solution* of this equation on the $\mathfrak{J}$.

If the operator-function M_t is $(L,0)$-bounded then we can gets the equation

$$\dot{f}(t) = M_t L_{t,1}^{-1} f(t)$$

with the operator-function $T_t = M_{t,1} L_{t,1}^{-1} \in C(\mathfrak{J}; \mathcal{L}(\mathfrak{F}^1))$. The solution for Cauchi problem $f(t_0) = f_0 \in \mathfrak{F}^1$ of this equation can be found [3] by the form $f(t) = \tilde{F}(t) f_0$ where *operator Cauchi*

$$\tilde{F}(t) = I_{\mathfrak{F}^1} + \int\limits_{t_0}^{t} T_{t_1} dt_1 + \sum_{n=2}^{\infty} \int\limits_{t_0}^{t} \int\limits_{t_0}^{t_n} \dots \int\limits_{t_0}^{t_2} T_{t_n} T_{t_{n-1}} \dots T_{t_1} dt_1 \dots dt_n \in \mathcal{L}(\mathfrak{F}^1).$$

Definition 3. The operator $U(t,\tau) = L_{t,1}^{-1} \tilde{F}(t) \tilde{F}^{-1}(\tau) L_{\tau,1} P_\tau$ is called an *evolution* (*solving*) *operator* for (1).

Theorem 3. [2] *The evolution operator has the following properties:*
(i) $U(t,t) = P_t$;
(ii) $U(t,s)U(s,\tau) = U(t,\tau)$;
(iii) $U(t,\tau)\Big|_{\mathfrak{U}^1_\tau} = \left[U(\tau,t)\Big|_{\mathfrak{U}^1_t} \right]^{-1}$;
(iv) $\|U(t,\tau)\|_{\mathcal{L}(\mathfrak{U})} \le K \exp\left(\int\limits_{\tau}^{t} \|T_s\|_{\mathcal{L}(\mathfrak{F}^1)} ds \right)$ $(\tau \le t)$.

References

[1] Sviridyuk G.A., Fedorov V.E. (2003) Linear Sobolev Type Equations and Degenerate Semigroups of Operators. Utrech, Boston, Koln, VSP.
[2] Sagadeeva M.A. (2012) The Solvability of Nonstationary Problem of Filtering Theory // Bulletin of the South Ural State University. Series "Mathematical Modelling, Programming & Computer Software", N No. 27 (286), Issue 13, pp. 86–98. (in Russian)
[3] Daletskiy Yu.L., Krein M.G. (1970) The Stability of Solutions for Differential Equations in Banach Spaces. Moscow, Science. (in Russian)