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An Alternative Approximation of the Degenerate Strongly Continuous Operator Semigroup

Inheriting and continuing the tradition, dating back to the Hill–Iosida–Feller–Phillips–Miyadera theorem, the new way of construction of the approximations for strongly continuous operator semigroups with kernels is suggested in the framework of the Sobolev type equations theory, which experiences an epoch of blossoming. We introduce the concept of relatively radial operator, containing the condition in the form of estimates for the derivatives of the relative resolvent. The existence of C_0-semigroup on some subspace of the original space is shown, the sufficient conditions of its coincidence with the whole space are given. The results are very useful in numerical study of different nonclassical mathematical models considered in the framework of the theory of the first order Sobolev type equations [1], and also to spread the ideas and methods to the higher order Sobolev type equations [2].

Let $\mathcal{U}$ and $\mathcal{F}$ be Banach spaces, operators $L \in \mathcal{L}(\mathcal{U};\mathcal{F})$ and $M \in \mathcal{C}l(\mathcal{U};\mathcal{F})$, function $f(\cdot) : \mathbb{R} \to \mathcal{F}$. Consider the Cauchy problem

$$u(0) = u_0 \tag{1}$$

for the operator-differential equation

$$L\,\dot{u} = Mu + f. \tag{2}$$

Following [1, 3], introduce the L-resolvent set $\rho^L(M) = \{\mu \in \mathbb{C} : (\mu L - M)^{-1} \in \mathcal{L}(\mathcal{F};\mathcal{U})\}$ and the L-spectrum $\sigma^L(M) = \overline{\mathbb{C}} \setminus \rho^L(M)$ of M. The operator functions $(\mu L - M)^{-1}$, $R^L_\mu(M) = (\mu L - M)^{-1}L$, $L^L_\mu(M) = L(\mu L - M)^{-1}$ are called L-resolvent, right and left L-resolvents of M.

Definition 1. The operator M is called *radial with respect to* L (shortly, L-radial), if
(i) $\exists a\in \mathbb{R} \quad \forall \mu > a \quad \mu\in \rho^L(M)$
(ii) $\exists K > 0 \quad \forall \mu > a \quad \forall n\in \mathbb{N}$

$$\max\{\|\frac{1}{n!}\frac{d^n}{d\mu^n}R^L_\mu(M)\|_{\mathcal{L}(\mathcal{U})}, \|\frac{1}{n!}\frac{d^n}{d\mu^n}L^L_\mu(M)\|_{\mathcal{L}(\mathcal{F})}\} \leq \frac{K}{(\mu-a)^{n+1}}$$

Remark 1. Without loss of generality one can put $a = 0$ in definiton 1.

Set $\mathcal{U}^0 = \ker L \quad \mathcal{F}^0 = \ker L^L_\mu(M)$. By $L_0 \quad (M_0)$ denote restriction of $L \quad (M)$ to lineal $\mathcal{U}^0 \quad (\mathrm{dom} M_0 = \mathcal{U}^0 \cap \mathrm{dom} M)$.

By $\mathcal{U}^1 \quad (\mathcal{F}^1)$ denote the closure of the lineal $\mathrm{im}\ R^L_\mu(M) \quad (\mathrm{im}\ L^L_\mu(M))$ by norm of $\mathcal{U} \quad (\mathcal{F})$.

By $\tilde{\mathcal{U}} \quad (\tilde{\mathcal{F}})$ denote the closure of the lineal $\mathcal{U}^0 \dot{+}\ \mathrm{im}\ R^L_\mu(M) \quad (\mathcal{F}^0 \dot{+}\ \mathrm{im}\ L^L_\mu(M))$ by norm of $\mathcal{U} \quad (\mathcal{F})$. Obviously, $\mathcal{U}^1 \quad (\mathcal{F}^1)$ is the subspace in $\tilde{\mathcal{U}} \quad (\tilde{\mathcal{F}})$.

Consider two equivalent forms of (2)

$$R^L_\alpha(M)\dot{u} = (\alpha L - M)^{-1} M u, \tag{3}$$

$$L^L_\alpha(M)\dot{f} = M(\alpha L - M)^{-1} f \tag{4}$$

as concrete interpretations of the equation

$$A\dot{v} = Bv, \tag{5}$$

defined on a Banach space $\mathcal{V}$, where the operators $A, B \in \mathcal{L}(\mathcal{V})$

Definition 2. The vector-function $v \in C(\overline{\mathbb{R}_+}; \mathcal{V})$, differentiable on $\mathbb{R}_+$ and satisfying (5) is called *a solution* of (5).

A little away from the standard [4], following [3] define

Definition 3. The mapping $V^\cdot \in C(\overline{\mathbb{R}_+}; \mathcal{L}(\mathcal{V}))$ is called *a semigroup of the resolving operators* (a resolving semigroup) of (5), if

(i) $V^s V^t v = V^{s+t} v$ for all $s, t \geq 0$ and any v from the space $\mathcal{V}$;

(ii) $v(t) = V^t v$ is a solution of the equation (5) for any v from a dense in $\mathcal{V}$ set.

The semigroup is called *uniformly bounded*, if

$$\exists C > 0 \quad \forall t \geq 0 \quad \|V^t\|_{\mathcal{L}(\mathcal{V})} \leq C.$$

Theorem 1. *Let M be L-radial. Then there exists a uniformly bounded and strongly continuous resolving semigroup of (3) ((4)), treated on the subspace $\tilde{\mathcal{U}}$ $(\tilde{\mathcal{F}})$, presented in the form:*

$$U^t = s - \lim_{k \to +\infty} \frac{(-1)^{k-1}}{(k-1)!} \left(\frac{k}{t}\right)^k \left(\frac{d^{k-1}}{d\mu^{k-1}} R^L_\mu(M)\right)\Bigg|_{\mu=\frac{k}{t}},$$

$$(F^t = s - \lim_{k \to +\infty} \frac{(-1)^{k-1}}{(k-1)!} \left(\frac{k}{t}\right)^k \left(\frac{d^{k-1}}{d\mu^{k-1}} L^L_\mu(M)\right)\Bigg|_{\mu=\frac{k}{t}}).$$

The semigroup $\tilde{U}^t$ $(\tilde{F}^t)$ at first is defined not on the whole space $\mathcal{U}$ $(\mathcal{F})$, but on some subspace $\tilde{\mathcal{U}}$ $(\tilde{\mathcal{F}})$. Introduce the sufficient condition of their coincidence: $\mathcal{U} = \tilde{\mathcal{U}}$ $(\mathcal{F} = \tilde{\mathcal{F}})$.

Theorem 2. [1] *Let the space $\mathcal{U}$ $(\mathcal{F})$ be reflexive, the operator M be L-radial. Then $\mathcal{U} = \mathcal{U}^0 \oplus \mathcal{U}^1 \quad (\mathcal{F} = \mathcal{F}^0 \oplus \mathcal{F}^1)$.*

References

[1] Sviridyuk G.A., Fedorov V.E. (2003) Linear Sobolev Type Equations and Degenerate Semigroups of Operators. Utrecht, Boston, Köln, Tokyo, VSP.

[2] Sviridyuk G.A., Zamyshlyaeva A.A. (2006) The Phase Spaces of a Class of Linear Higher-order Sobolev Type Equations // Differential Equations, vol. 42, no. 2. pp. 269–278.

[3] Sviridyuk G.A. (1994) Linear Sobolev Type Equations and Strongly Continuous Semigroups of the Resolving Operators with Kernels // Doklady akademii nauk, vol. 337, no. 5, pp. 581–584.

[4] Hille E., Phillips R.S. (1957) Functional Analysis and Semi-Groups. American Mathematical Society, Providence, Rhode Island.