Inverse problem for a degenerate evolution equation with overdetermination on the solution semigroup kernel

The inverse problem for a linearized quasi-stationary phase field model is explored. This problem is reduced to a linear inverse problem for the first order differential equation in a Banach space with a degenerate operator at the derivative and an overdetermination condition on the solution semigroup kernel. The theorem on unique solvability for the inverse problem is obtained by virtue of the theory degenerate operator semigroups methods as in a nonlinear inverse problem for a hydrodynamical equations systems was researched.

Let \( \Omega \subset \mathbb{R}^n \) be a bounded domain with a smooth boundary \( \partial \Omega \), \( T > 0 \), \( \beta, \delta \in \mathbb{R} \). Consider the initial-boundary value problem

\[
(\beta + \Delta)(v(x,0) - v_0(x)) = 0, \quad x \in \Omega, \quad (1)
\]

\[
(1 - \delta)v + \delta \frac{\partial v}{\partial n}(x,t) = (1 - \delta)w + \delta \frac{\partial w}{\partial n}(x,t) = 0, \quad (x,t) \in \partial \Omega \times [0, T], \quad (2)
\]

for the system of equations

\[
v_t(x,t) = \Delta v(x,t) - \Delta w(x,t) + b_1(x,t)u(t), \quad (x,t) \in \Omega \times [0, T], \quad (3)
\]

\[
0 = v + (\beta + \Delta)w + b_2(x,t)u(t), \quad (x,t) \in \Omega \times [0, T], \quad (4)
\]

with overdetermination condition on the subspace of degeneracy

\[
\int_{\Omega} K(y)w(y,t)dy = \psi(t), \quad (x,t) \in \Omega \times [0, T]. \quad (5)
\]

Up to a linear change of functions \( v(x,t), w(x,t) \), the system coincides with the linearization of the quasistationary phase-field model \([3]\), describing phase transitions of the first kind in terms of the mesoscopic theory. The unknown functions of the inverse problem (1)–(5) are \( v(x,t), w(x,t), u(t) \).

Denote \( Aw = \Delta w, D_A = H^2_\delta(\Omega) \subset L_2(\Omega), \langle \cdot, \cdot \rangle \) is inner product in \( L_2(\Omega) \). Let \( \{ \varphi_k : k \in \mathbb{N} \} \) be orthonormal in \( L_2(\Omega) \) eigenfunctions of the operator \( A \), enumerated with respect to the nonascending order of the eigenvalues \( \{ \lambda_k : k \in \mathbb{N} \} \), counting their multiplicities.

**Theorem 1.** Let \( -\beta \in \sigma(A), b_i \in C^1([0, T]; L_2(\Omega)), i = 1, 2, \) and \( \langle b_1(\cdot, t), \varphi_k \rangle = 0 \) for \( \lambda_k \neq -\beta, K \in L_2(\Omega), \langle K, \varphi_k \rangle = 0 \) for \( \lambda_k = -\beta, \langle K, b_2(\cdot, t) \rangle \neq 0 \) for all \( t \in [0, T] \), \( \psi \in C^1[0, T], v_0 \in H^2_\delta(\Omega) \). Then there exists a unique solution of the problem (1)–(5).
References