Dimension and Lyapunov exponents in conformal non-hyperbolic dynamics

3. Lyapunov spectrum of expansive Markov interval maps

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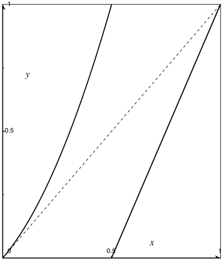
Będlewo, 2013

Expansive Markov system

$T : \Lambda \to \Lambda$ $C^{1+\varepsilon}$ expansive conformal Markov repeller, tempered distortion
$C_1, \ldots, C_k \subset [0,1]$ essentially disjoint intervals,
$T : C_1 \cup \ldots \cup C_k \to [0,1]$ having Markov property, $T|_\Lambda$ semi-conj to $\sigma|_{\Sigma_A^+}$



Aim to study level sets of Lyapunov exponents

$$\mathscr{L}(\alpha) = \{x \in \Lambda \colon \lambda(x) \stackrel{\text{def}}{=} \lim_{n\to\infty} \frac{1}{n} \log |(T^n)'(x)| = \alpha\}.$$

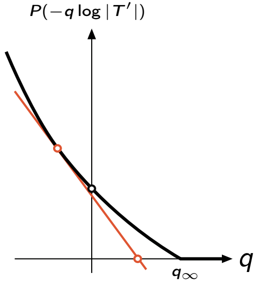
Expansive Markov system:

Topological pressure – geometric potential – Legendre-Fenchel transform

Given $\alpha \neq 0$ define Legendre-Fenchel transform of $q \mapsto P(-q \log |T'|)$

$$E(\alpha) \stackrel{\text{def}}{=} \inf_{q \in \mathbb{R}} \big(\alpha\, q + P(-q \log |T'|)\big),$$

as well as $$F(\alpha) \stackrel{\text{def}}{=} \frac{E(\alpha)}{\alpha}, \quad F(0) \stackrel{\text{def}}{=} \lim_{\alpha \to 0+} \frac{E(\alpha)}{\alpha}$$



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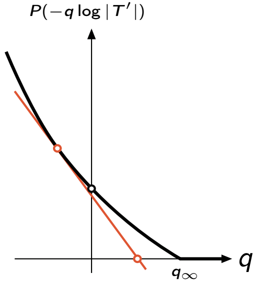

- $q_\infty \le \dim_{\mathrm{H}} \Lambda$
  hyperbolic dimension
- for $\alpha = -\frac{d}{dq} P'(-q\log|T'|)$
  $$F(\alpha) = \frac{1}{\alpha}(\alpha\, q + P(q))$$
  $$0 = P(q) - \alpha(F(\alpha) - q)$$
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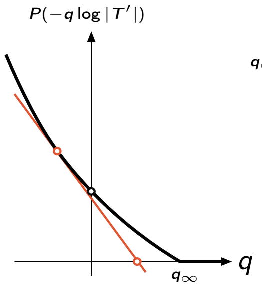

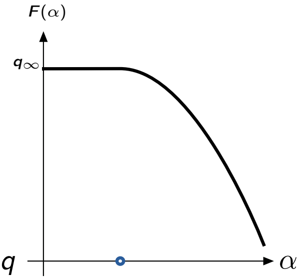

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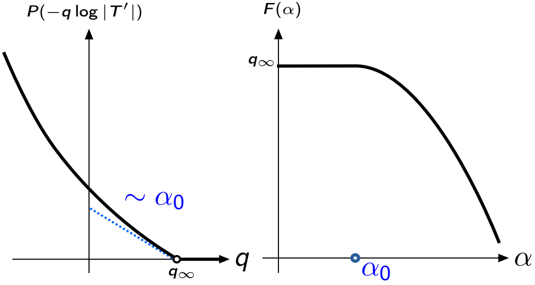

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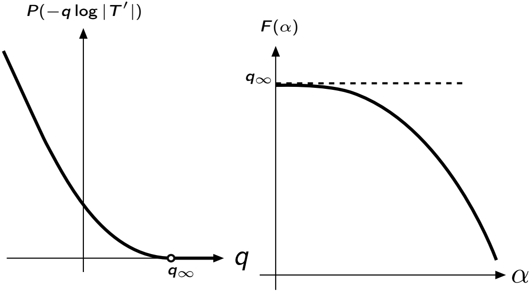

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Conformal measures

[Patterson '76, Sullivan '83, Denker–Urbański '91, Walters '78, Yuri '99]
Given $\psi \in C(\Lambda)$ consider the transfer operator $\mathcal{L}_\psi \colon C(\Lambda) \to C(\Lambda)$ defined by

$$(\mathcal{L}_\phi g)(x) \stackrel{\text{def}}{=} \sum_{Ty=x} g(y) e^{\psi(y)}.$$

Let $\mathcal{L}_\psi^*$ be the dual of $\mathcal{L}_\psi$. The map $\mu \mapsto \mathcal{L}_\psi^* \mu \, (\mathcal{L}_\psi^* \mu(1))^{-1}$ has a fixed point $\mu_\psi \in \mathcal{M}(\Lambda)$. Let $\lambda_\psi = \mathcal{L}_\psi^* \mu_\psi(1)$.

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The measure μ_ψ is $e^{P(\psi)-\psi}$-conformal in the sense that for every special set A a.e.

$$\frac{d(\mu_\psi \circ T)|_A}{d\mu_\psi|_A} = \lambda_\psi e^{-\psi} \quad \text{and } \lambda_\psi = e^{P(\psi)}.$$

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Observe that for a T^n-special set

$$(\mu_\psi \circ T^n)(A) = \int_A e^{nP(\psi) - S_n\psi} \, d\mu_\psi.$$

Conformal measures on cylinders

Given $q \le q_\infty$ study $e^{P(-q\log|T'|)+q\log|T'|}$-conformal measure μ_q

$$(\mu_q \circ T^n)(C_n(x)) = \int_{C_n(x)} e^{n\,P(-q\log|T'|)+q\log|(T^n)'|}\,d\mu_q.$$

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Lemma (Tempered distortion)

There exists $\rho_n \to 0$ *such that* $\max\limits_{(\omega_1\ldots\omega_n)} \max\limits_{x,y\in C(\omega_1\ldots\omega_n)} \log \frac{|(T^n)'(x)|}{|(T^n)'(y)|} \leq n\rho_n$.

Proof.

$$\log \max_{(\omega_1\ldots\omega_n)} \max_{x,y\in C(\omega_1\ldots\omega_n)} \frac{|(T^n)'(x)|}{|(T^n)'(y)|} \leq \mathrm{C}^{\mathrm{st}} \sum_{k=1}^{n} \max_x |C_k(x)|^\varepsilon$$

□

Hence, up to a subexponential factor, for all $x \in \Lambda$

$$\mu_q(C_n(x)) \asymp e^{-n\,P(-q\log|T'|)}\,|(T^n)'(x)|^{-q}.$$

Multifractal analysis – Lyapunov exponents:

weak multifractal formalism – upper bound for dimension

Proposition

For every $\alpha > 0$

$$\dim_{\mathrm{H}} \mathscr{L}(\alpha) \le F(\alpha).$$

Proof.

Let $x \in \Lambda$ with $\lambda(x) = \alpha$

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Recalling

$$\mu_q(C_n(x)) \asymp e^{-n\,P(-q\log|T'|)}\,|(T^n)'(x)|^{-q},$$

hence

$$\underline{d}_{\mu_q}(x) \sim \frac{\log \mu_q(C_n(x))}{\log|C_n(x)|} \sim \frac{\alpha\, q + P(-q\log|T'|)}{\alpha}$$

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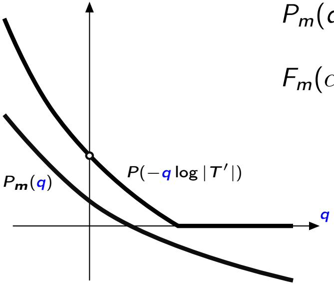
By the Mass Distribution Principle $\dim_{\mathrm{H}} \mathscr{L}(\alpha) \leq \frac{1}{\alpha}(\alpha\, q + P(-q\log|T'|))$. □

Multifractal analysis – Lyapunov exponents:

weak multifractal formalism – approximation by mixing expanding repellers

Consider $\Lambda_1 \subset \ldots \subset \Lambda_m \subset \Lambda$ mixing expanding repeller, $\Lambda_m \to \Lambda$ in Hausdorff topology. Each Λ_m possesses q-conformal ν_q^m.
All Λ_m have tempered distortion with *very same* $(\rho_n)_n$, and each Λ_m has bounded distortion.



$$P_m(q) \stackrel{\text{def}}{=} P_{T|\Lambda_m}(-q \log|T'|)$$

$$F_m(\alpha) \stackrel{\text{def}}{=} \frac{1}{\alpha} \inf_{q\in\mathbb{R}} (\alpha\, q + P_m(q)) = \frac{1}{\alpha} E_m(\alpha)$$

$$\underline{\alpha}_m \stackrel{\text{def}}{=} \inf_{\nu\in\mathcal{M}(\Lambda_m)} \lambda(\nu), \quad \overline{\alpha}_m \stackrel{\text{def}}{=} \sup_{\nu\in\mathcal{M}(\Lambda_m)} \lambda(\nu)$$

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Lemma

1) $\lim_{m\to\infty} P_m = P$ *pointwise*
2) $\lim_{m\to\infty} F_m = F$ *pointwise*
3) $\lim_{m\to\infty} \underline{\alpha}_m = \underline{\alpha}$, $\lim_{m\to\infty} \overline{\alpha}_m = \overline{\alpha}$

Proof. 1).

$P_m(q)$ non-decreasing. Suppose $\delta \stackrel{\text{def}}{=} P(q) - \sup_m P_m(q) > 0$. For all n, up to a subexponential factor, we have

$$\mu_q(C_{\omega_1\dots\omega_n})e^{nP(q)} \asymp |(T^n)'(x)|^{-q} \asymp \mu_q^m(C_{\omega_1\dots\omega_n})e^{nP_m(q)}$$

For n large, $2\rho_n < \delta$. Then $\lim_{m\to\infty} \mu_q^m$ cannot be probability measure. $\Rightarrow\Leftarrow$ □

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Proof. 2).

(P_m, E_m) Legendre-Fenchel pair, hence

$$E = \lim_{m\to\infty} E_m \quad \text{iff} \quad P = \lim_{m\to\infty} P_m$$

pointwise. Hence, $\lim_m F_m(\alpha) = \frac{1}{\alpha} \lim_m E_m(\alpha) = \frac{1}{\alpha} E(\alpha) = F(\alpha)$. □

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Proof. 3).

Observe that $\sup_m \overline{\alpha}_m \leq \overline{\alpha}$. On the other hand,

$$\overline{\alpha} = \sup \lambda(\nu) = \lim_{q\to-\infty} \frac{-1}{q} \sup_{\nu} \left(h(\nu) - q\lambda(\nu)\right) = \lim_{q\to-\infty} \frac{-1}{q} P(-q \log |T'|) .$$

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Further, if $q < 0$ then $P_m(q) + q\,\overline{\alpha}_m \leq P_m(0) \leq P(0)$ and hence

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Hence, $\overline{\alpha} \leq \sup_m \overline{\alpha}_m$. □

Multifractal analysis – Lyapunov exponents:

weak multifractal formalism – interior of spectrum

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Proof.

Remains to show $\geq$.
We have $\alpha \in (\underline{\alpha}_m, \overline{\alpha}_m)$ for $m \geq 1$ sufficiently large. By [Pesin-Weiss]

$$\dim_{\mathrm{H}} \mathscr{L}(\alpha) \geq \dim_{\mathrm{H}} \mathscr{L}(\alpha) \cap \Lambda_m = F_m(\alpha).$$

$\dim_{\mathrm{H}} \mathscr{L}(\alpha) \geq F(\alpha)$ follows from $F(\alpha) = \lim_{m\to\infty} F_m(\alpha)$. □

Multifractal analysis – Lyapunov exponents:

weak multifractal formalism – boundary of spectrum

To study the case $\alpha \in \{\underline{\alpha}, \overline{\alpha}\}$ we construct "bridging measures".

Similar ideas are contained in:

- [Besicovich '34] sum of digits of reals represented in dyadic system
- [Barreira, Schmeling '00] "non-typical" points have full entropy / dimension
- [Takens, Verbitsky '03] variational principle for the topological entropy of certain non-compact sets
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Plan: Construct Borel probability measure μ with $\mu(\mathscr{L}(\alpha)) > 0$, calculate its local dimension, apply Mass Distribution Principle.

Bridging measure

Let $\Sigma_{A_1} \subset \Sigma_{A_2} \subset \ldots \subset \Sigma$ family of mixing SFT's

μ_ℓ equilibrium states w.r.t. $\sigma|_{\Sigma_{A_\ell}}$ and potentials ϕ_ℓ with $P_\ell(\phi_\ell) = 0$

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Fix increasing sequence $m_\ell \to \infty$. On each $\omega_{m_1} \in \Sigma^{m_1}$ put

$$\mu([\omega_{m_1}]) = \mu_1([\omega_{m_1}])$$

For any $\ell \geq 2$ sub-distribute μ on $[\omega_{m_\ell}] = \bigcup_{\tau_{m_{\ell+1}-m_\ell}} [\omega_{m_\ell}\tau_{m_{\ell+1}-m_\ell}]$ as

$$\mu([\omega_{m_\ell}\tau_{m_{\ell+1}-m_\ell}]) = \mu([\omega_{m_\ell}])\, \mu_{\ell+1}([\tau_{m_{\ell+1}-m_\ell}]) \cdot N_{\ell+1}([\omega_{m_\ell}])$$

where $N_{\ell+1}([\omega_{m_\ell}])$ is the normalizing constant

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For any $\ell \geq 2$ sub-distribute μ on $[\omega_{m_\ell}] = \bigcup_{\tau_{m_{\ell+1}-m_\ell}} [\omega_{m_\ell} \tau_{m_{\ell+1}-m_\ell}]$ as

$$\mu([\omega_{m_\ell} \tau_{m_{\ell+1}-m_\ell}]) = \mu([\omega_{m_\ell}])\, \mu_{\ell+1}([\tau_{m_{\ell+1}-m_\ell}]) \cdot N_{\ell+1}([\omega_{m_\ell}])$$

where $N_{\ell+1}([\omega_{m_\ell}])$ is the normalizing constant
For each ℓ probability on $\Sigma_{A_\ell}^{m_\ell}$ is well-defined on cylinders. Extend it to Σ.

Lemma

$\mu([\omega\,\tau]) \asymp_\ell \mu([\omega])\, \mu_{\ell+1}([\tau])$ *whenever* $\omega\,\tau = \omega_{m_\ell}\, \tau_{m_{\ell+1}-m_\ell}$.

Bridging measure – properties

Assume that the following sequences converge

$$h_\ell = h(\mu_\ell), \quad \lambda_\ell = \lambda(\mu_\ell), \quad d_\ell = \frac{h_\ell}{\lambda_\ell} = \dim_{\mathrm{H}} \mu_\ell.$$

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$$H_m^\mu(x) = -\frac{1}{m} \log \mu(C_m(x)), \quad L_m(x) = \frac{1}{m} \log |(T^m)'(x)|.$$

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Proposition

If $(m_\ell)_\ell$ increases sufficiently fast, then for μ-almost every $x \in \Lambda$

$$\lim_{m\to\infty} H_m^\mu(x) = \lim_{\ell\to\infty} h_\ell, \quad \lim_{m\to\infty} L_m(x) = \lim_{\ell\to\infty} \lambda_\ell, \quad \underline{d}_\mu(x) \geq \lim_{\ell\to\infty} d_\ell.$$

Hence, for $\alpha = \lim_{\ell\to\infty} \lambda_\ell$ we have

$$\dim_{\mathrm{H}} \mathscr{L}(\alpha) \geq \liminf_{\ell\to\infty} d_\ell.$$

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Proof of Proposition

Lemma

For ε, δ *there exists* $M \geq 1$ *such that for* $m_\ell \geq M$ *we have* $\mu(B_\ell) \leq \delta$*, where*

$$B_\ell = \left\{ x \colon \left| m\, L_m(x) - \left(m_\ell L_{m_\ell}(x) + (m - m_\ell)\lambda_\ell \right) \right| > m\,\varepsilon \text{ for some } m_\ell < m \leq m_{\ell+1} \right\}.$$

Proof.

$m_\ell L_{m_\ell}(x) + (m - m_\ell) L_{m-m_\ell}(T^{m_\ell}(x)) = m\, L_m(x) \to m\,\lambda_\ell = m_\ell\,\lambda_\ell + (m - m_\ell)\lambda_\ell$ uniformly on set of measure μ_ℓ at least $1 - \delta$. For each cylinder $[\omega] = [\omega_{m_\ell}]$

$$\mu(B_\ell \cap [\omega]) \leq \sum_\tau \mu([\omega\, \tau_{m_{\ell+1} - m_\ell}]) \asymp \sum_\tau \mu([\omega])\, \mu_\ell([\tau_{m_{\ell+1} - m_\ell}]) \leq \mu([\omega])\, \delta$$

Hence, summing over all $\omega = \omega_{m_\ell}$ we obtain $\mu(B_\ell) \leq \delta$. □

Bridging measure – properties

For $(\varepsilon_\ell)_\ell$ and for $(\delta_\ell)_\ell$ summable, Borel-Cantelli implies $\mu(\limsup_\ell B_\ell) = 0$.

Hence, for μ-almost every x for sufficiently large ℓ

$$\left|mL_m(x) - \left(m_\ell L_{m_\ell}(x) + (m - m_\ell)\lambda_\ell\right)\right| \le m\,\varepsilon_\ell \qquad \text{for every } m = m_\ell + 1, \ldots, m_{\ell+1}$$

hence, choosing $(\varepsilon_\ell)_\ell$ appropriately, for every such m

$$L_m(x) \sim \frac{m_\ell\,\lambda_\ell + (m - m_\ell)\,\lambda_{\ell+1}}{m} \quad \Longrightarrow \quad \lambda(x) = \lim_{\ell\to\infty} \lambda_\ell.$$

So μ-a.e. $x \in \mathscr{L}(\alpha)$. Similar, the 'finite-level' entropy for μ-a.e. x satisfies

$$H^\mu_m(x) \sim \frac{m_\ell\,h_\ell + (m - m_\ell)\,h_{\ell+1}}{m} \quad \Longrightarrow \quad \lim_{m\to\infty} H^\mu_m(x) = \lim_{\ell\to\infty} h_\ell.$$

Thus, up to some constant (tempered distortion)

$$\underline{d}_\mu(x) \sim \frac{\log\mu(C_{m_\ell}(x))}{\log|C_{m_\ell}(x)|}$$

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By the Mass Distribution Principle, $\dim_{\mathrm{H}} \mathscr{L}(\alpha) \ge \liminf d_\ell$. **(Proposition)** □

Multifractal analysis – Lyapunov exponents

$T : \Lambda \to \Lambda$ $C^{1+\varepsilon}$ expansive conformal Markov repeller, tempered distortion
Suppose there exist $\Lambda_1 \subset \ldots \subset \Lambda_m \subset \Lambda$ mixing expanding repeller, $\Lambda_m \to \Lambda$ in Hausdorff topology. All Λ_m have tempered distortion with very same $(\rho_n)_n$, and each Λ_m has bounded distortion.

Theorem (G-Rams '09)

Then for every $\alpha \in (\underline{\alpha}, \overline{\alpha})$ *we have* $\dim_{\mathrm{H}} \mathscr{L}(\alpha) = F(\alpha)$. *If* $\mathscr{L}(0) \neq \emptyset$ *then* $\dim_{\mathrm{H}} \mathscr{L}(0) = \dim_{\mathrm{H}} \Lambda \geq q_\infty = F(0)$.