Anisotropy 2014

List of talks

Long talks

A Priori Estimate for Fractional Nonlinear Degenerate Diffusion Equations on bounded domains

Matteo Bonforte, Univesidad Autonoma de Madrid

We investigate quantitative properties of nonnegative solutions $u(t,x) \geq 0$ to the nonlinear fractional diffusion equation,

$$\partial_t u + \mathcal{L}(u^m) = 0,$$

posed in a bounded domain, $x \in \Omega \subset \mathbb{R}^N$ for $t > 0$ and $m > 1$. As $\mathcal{L}$ we can take the most common definitions of the fractional Laplacian $(-\Delta)^s$, $0 < s < 1$, in a bounded domain with zero Dirichlet boundary conditions, as well as more general classes of operators. We consider a class of very weak solutions for the equation at hand, that we call weak dual solutions, and we obtain a priori estimates in the form of smoothing effects, absolute upper bounds, lower bounds, and Harnack inequalities. We also investigate the boundary behaviour and we obtain sharp estimates from above and below. The standard Laplacian case $s = 1$ or the linear case $m = 1$ are recovered as limits. The method is quite general, suitable to be applied to a number of similar problems that will be briefly discussed as examples.

As a consequence, we can prove existence and uniqueness of minimal weak dual solutions with data in $L^1_{\Phi_1}$, where Φ_1 is the first eigenfunction of $\mathcal{L}$. We also briefly show existence and uniqueness of H^{-s} solutions with a different approach. As a byproduct, we derive similar estimates for the elliptic semilinear equation

$$\mathcal{L}S^m = S$$

and we prove existence and uniqueness of $H^{-s}(\Omega)$ solutions via parabolic techniques. Solutions to this elliptic problem represents the asymptotic profiles of the rescaled solutions, namely the stationary states of the rescaled equation $\partial_t v = -\mathcal{L}(v^m) + v$.

Finally, we will study the asymptotic behaviour. We will prove sharp rates of decay of the rescaled solution to the unique stationary profile S and also for the relative error $v/S - 1$. The sharp rates of convergence can be obtained with two different methods: one is based on the above estimates, that guarantee existence of the "friendly giant". Another approach is given by a new entropy method, based on the so-called Caffarelli-Silvestre extension.

This is a joint work with J. L. Vázquez (UAM, Madrid, Spain) and Y. Sire (Univ. Marseille, France).

References

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[BSV] M. B., Y. Sire, J. L. Vázquez, Existence, Uniqueness and Asymptotic behaviour for fractional porous medium equations on bounded domains. To appear in DCDS (2014). http://arxiv.org/abs/1404.6195

[BV2] M. B., J. L. Vázquez, Nonlinear Degenerate Diffusion Equations on bounded domains with Restricted Fractional Laplacian. In Preparation (2014).

Optical Flow on Moving Manifolds

Markus Grasmair, Norges teknisk-naturvitenskapelige universitet

Optical flow is a powerful tool for the study and analysis of motion in a sequence of images. In this talk we discuss a Horn–Schunck type spatio-temporal regularization functional for image sequences that have a non-Euclidean, time varying image domain. The motivation for this is an application in volumetric microscopy, where one studies the early development of a zebra-fish embryo moving on the surface of the embryo's yolk cell. First we construct a Riemannian metric that describes the deformation and structure of this evolving surface. The resulting functional can be seen as natural geometric generalization of previous work by Weickert and Schnörr (2001) and Lefèvre and Baillet (2008) for static image domains. We show the existence and wellposedness of the corresponding optical flow problem and derive necessary and sufficient optimality conditions, and we show numerical results demonstrating the functionality of this approach. This is joint work with Martin Bauer and Clemens Kirisits (University of Vienna).

The Euler-Lagrange equation for the anisotropic least gradient problem

José M. Mazón, Universitat de València

In [JMN], motivated by the Conductivity Imaging Problem, the following general least gradient problem is studied:

$$\inf \left\{ \int_\Omega \phi(x, Du) \colon u \in BV(\Omega),\ u|_{\partial\Omega} = f \right\}, \tag{1}$$

where ϕ a metric integrand and $f \in C(\partial\Omega)$.

We used the relaxed functional associated with the functional in (1), to get the Euler-Lagrange equation associated with this variational problem.

The particular case $\phi(x, \xi) = |\xi|$ corresponds to the classical least gradient problem, for which Sternberg, Williams and Ziemer in [SWZ] proved that for $\Omega \subset \mathbb{R}$ that has non-negative mean curvature (in a weak sense) and is not locally area-minimizing and $f \in C(\partial\Omega)$, there exists a unique function of least gradient $u \in BV(\Omega) \cap C(\overline{\Omega})$ such that $u = f$ on $\partial\Omega$. The relaxed energy functional and the corresponding Euler-Lagrange equation has been studied in [MRS].

References

[JMN] R.L. Jerrard, A. Moradifam and A.I. Nachman, Existence and uniqueness of minimizers of general least gradient problems. Prepint. arXiv: 1305.0535v1.

[MRS] J.M. Mazón, J.D. Rossi and S. Segura de León, Functions of Least Gradient and 1-Harmonic functions, *Indiana Univ. J. Math.* **63** (2014), 1067-1084.

[SWZ] P. Sternberg, G. Williams and W.P. Ziemer, Existence, uniqueness, and regularity for functions of least gradient. *J. Reine Angew. Math.* **430** (1992) 35–60.

Existence of solutions to the Kobayashi-Warren-Carter system

Salvador Moll, Universitat de València

In this talk, I will present some recent results in collaboration with Ken Shirakawa (University of Chiba) and Hiroshi Watanabe (Salesian Polytechnic, Tokyo) about existence of solutions, energy-dissipation and large time behavior for a system of two parabolic PDEs modeling the grain boundary motion of a 2-D polycrystal.

Anisotropy in parabolic equations with non-standard growth conditions

Agnieszka Świerczewska-Gwiazda, Uniwersytet Warszawski

The talk will be dedicated to an abstract parabolic problem. We present a generalization of standard Leray-Lions operators – instead of considering polynomial growth conditions of the highest order nonlinear term we allow for anisotropic and space inhomogeneous conditions. In a consequence the problem has solutions in generalized Orlicz spaces.

Numerical Solutions of Fractional Stefan Problems

Vaughan R. Voller, University of Minnesota

The Stefan problem, involving the tracking of a phase change interface, provides the framework for studying moving boundary problems in a variety of physical situations. The solution of the classical one-dimensional Stefan problem predicts that in time t the phase interface goes as $s(t) \sim t^{\frac{1}{2}}$. In the presence of heterogeneity, however, anomalous behavior can be observed where the time exponent $n \neq \frac{1}{2}$. In such a case, it may be appropriate to write down the governing equations of the Stefan problem in terms of fractional order time ($1 \geq \beta > 0$) and space ($1 \geq \alpha > 0$) derivatives.

In this talk we start with providing a physical justification for using fractional Stefan problems and introduce a limit case problem related to the horizontal moisture movement in a porous medium. Two model forms are presented; one assuming a sharp interface between the advancing wet and dry domains, the other assuming a diffuse interface, of finite thickness, across which a liquid fraction changes smoothly from a value of one to a value of zero. Analytical and numerical solutions are provided for both forms and we illustrate and discuss two important features. The first relating to how to choose the direction over which non-locality operates, i.e., what is the difference between taking non-local contribution up-stream as opposed to down-stream of the interface. The second based on the observation that, in the fractional time case ($\beta < 1$), a solution of the fractional diffuse interface model in the sharp interface limit does not coincide with the solution of the sharp interface counterpart; negating a well known result for integer derivative Stefan problems.

Short talks

Fine analysis of solutions to a very singular strictly parabolic equation

Michał Łasica, Uniwersytet Warszawski

We consider a singular parabolic equation of form

$$u_t = u_{xx} + (\operatorname{sgn} u_x)_x.$$

Solutions to this kind of equations exhibit competition between smoothing due to one-dimensional Laplace operator and tendency to create flat facets due to singular operator $(\operatorname{sgn} u_x)_x$. We present results concerning analysis of qualitative behaviour and regularity of solutions. Our main result states that locally, the evolution is described by a system of free boundary problems for u in intervals between facets coupled with equations of evolution of facets. This leads to additional regularity of solutions.

A remark about the Lavrentiev gap phenomenon

Katarzyna Mazowiecka, Uniwersytet Warszawski

Minimizing harmonic maps (i.e. minimizers of the Dirichlet integral) with prescribed boudary conditions from the ball to the sphere may have singularities. For some boundary data it is known that all minimizers of the energy have singularities and the energy is strictly smaller than the infimum of the energy among the continuous extensions (the so called Lavrentiev gap phenomenon occurs). We prove that the Lavrentiev gap phenomenon for harmonic maps into spheres holds on a dense set of zero degree boundary data.

Solvability of the heat equation in a difficult way

Piotr B. Mucha, Uniwersytet Warszawski

I would like to present an approach of solving mono-dimensional systems on the example of the heat equations. The main idea is to replace the original equation by the nonlinear system of the form

$$u_t - \frac{d}{dx} L^N(u_x) = 0,$$

where $L^N(.)$ is a step function being an approximation of the identity.

As a result we obtain that the solution to the heat equation can be approximated by broken lines which are solutions to the nonlinear approximation. The key point is the evolution of the broken lines is easily and locally determined. I would like to explain why such approach may be efficient in more complex situations.

Duality based algorithm for solving the regularized anisotropic total variation flow

Monika Muszkieta, Politechnika Wrocławska

On general existence results for one-dimensional singular diffusion equations with spatially inhomogeneous driving force

Atsushi Nakayasu, University of Tokyo

We study the very singular diffusion equation in one space dimension with a general external force term. This problem is motivated by a crystalline curvature motion of graphs under spatially inhomogeneous conditions. In this talk we show some stability and existence results of a viscosity solution to the singular diffusion equation with a continuous periodic initial condition. This talk is based on the joint work with M.-H. Giga and Y. Giga.

Numerical solution to anisotropic diffusion equations in special cases

Łukasz Skonieczny, Uniwersytet Warszawski