Operator equations and domain dependence

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Abstract.

We study operator equations generalizing the chain rule for the second derivative,

$$(f \circ g)'' = f'' \circ g \cdot g'^2 + f' \circ g \cdot g'' \ , \ f, g \in C^2(\mathbb{R}) :$$

For $k \in \mathbb{N}$, consider operators $T : C^k(\mathbb{R}) \to C(\mathbb{R})$ and $A_1, A_2 : C^{k-1}(\mathbb{R}) \to C(\mathbb{R})$ such that

$$T(f \circ g) = (Tf) \circ g \cdot A_1 g + (A_2 f) \circ g \cdot Tg \quad , \quad f, g \in C^k(\mathbb{R}) \tag{1}$$

holds. Under mild assumptions on (T, A_1, A_2), which imply that (A_1, A_2) are rather different from T, the operators (T, A_1, A_2) are closely related and locally defined. Further, the natural domains are $C^k(\mathbb{R})$ for $k \in \{1, 2, 3\}$ since no solutions exist which would depend on $f^{(l)}$ for $l \geq 4$. For $k \in \{1, 2, 3\}$, the solutions are $A_2 f = |f'|^p \ \{\operatorname{sgn} f'\}$, $A_1 f = f'^{k-1} \cdot A_2 f$ with $p \geq k - 1$ and

$$Tf = [c \ S_k f + (H \circ f' \cdot f'^{k-1} - H) \cdot f'^{k-1}] \cdot |f'|^{p-k+1} \ \{\operatorname{sgn} f'\} \ ,$$

where the term $\{\operatorname{sgn} f'\}$ may be present or not, H is a continuous function on $\mathbb{R}$, $c \in \mathbb{R}$ is a constant and the differential operators S_k are given by

$$S_1 f = \ln |f'| \quad , \quad S_2 f = f'' \quad , \quad S_3 f = f' f''' - \frac{3}{2} f''^2 \ .$$

The solution $S_3 f$ is just f'^2 times the Schwarzian derivative of f (if $f \neq 0$). For $k = 1$, there are two further solutions of a slightly different type. Depending on S_k, the natural domain for T is $C^k(\mathbb{R})$ with $k \in \{1, 2, 3\}$. The natural domain for A_1, A_2 in all cases is $C^1(\mathbb{R})$. This is joint work with V. Milman.