On nuclearity of operators with s-nuclear adjoints

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We discuss the problems around a question, posed by A. Hinrichs and A. Pietsch (Math. Nachr. 283, No. 2 (2010), 232–261): Suppose T is an operator acting between Banach spaces X and Y, and let $s \in (0,1)$. Is it true that if T^* is s-nuclear then T is s-nuclear too?

As is well known, for $s = 1$, a negative answer was obtained already by T. Figiel and W.B. Johnson in 1973. The following result (which is sharp in the scale of s-nuclear operators in the sense of Theorem 2 below) gives one of the possible positive answers in this direction. To formulate the theorem, we need a definition: Let $0 < q \leq \infty$ and $1/s = 1/q + 1$. We say that X has the approximation property of order s, if for every $(x_n) \in l_q(X)$ (where $l_q(X)$ means $c_0(X)$ for $q = \infty$) and for every $\varepsilon > 0$ there exists a finite rank operator R in X such that $\sup_n ||Rx_n - x_n|| \leq \varepsilon$.

Theorem 1. If $s \in [2/3, 1]$ and T is a linear operator with s-nuclear adjoint from a Banach space X to a Banach space Y and if one of the spaces X^* or Y^{***} has the approximation property of order s, then the operator T is nuclear.

Remark: In the case where $s = 2/3$, a famous theorem due to A. Grothendieck says that every Banach space has the corresponding approximation property and the result is trivial. The case where $s = 1$ (Grothendieck's AP) was firstly investigated by Eve Oja and the author (C. R. Acad. Sc. Paris, Serie I, 305 (1987), 121–122).

The examples in the following result show that the condition "X^* or Y^{***} has the approximation property of order s" is essential.

Theorem 2. For each $s \in (2/3, 1]$ there exist a Banach space Z_s and a non-nuclear operator $T_s : Z_s^{**} \to Z_s$ so that Z_s^{**} has the metric approximation property, Z_s^{***} has the AP_r for every $r \in (0, s)$ and T_s^* is s-nuclear.

Remark: The space Z_1^{***} is isomorphic to a space of type $Z_1^* \oplus E$, where E is an asymptotically Hilbertian space. This gives us one more example of an asymptotically Hilbertian space which fails the approximation property.

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