Almost Transitive and Maximal Norms in Banach Spaces (S. J. Dilworth and B. Randrianantoanina)
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Almost Transitivity and Related Notions

Definition

Let $(X, \|\cdot\|)$ be a Banach space.

- $\text{Isom}(X, \|\cdot\|)$ is the group of all surjective linear isometries of X.
- $\text{GL}(X, \|\cdot\|)$ is the group of all surjective linear isomorphisms of X.

Definition

$(X, \|\cdot\|)$ is transitive if $\forall x, y \in S_X \ \exists \ T \in \mathrm{Isom}(X, \|\cdot\|)$ s.t. $T(x) = y$.

Problem (Banach-Mazur Rotation Problem)

Let $(X, \|\cdot\|)$ be a separable transitive Banach space. Is X isometrically isomorphic to a Hilbert space?

Several relaxed notions of transitivity have been defined:

- $(X, \|\cdot\|)$ is almost transitive if $\forall x, y \in S_X$ and $\varepsilon > 0$ $\exists T \in \mathrm{Isom}(X, \|\cdot\|)$ s.t. $\|T(x) - y\| < \varepsilon$.
- $(X, \|\cdot\|)$ is maximal (Pełczyński-Rolewicz, 1962) if there is no equivalent norm $|||\cdot|||$ such that $\mathrm{Isom}(X, \|\cdot\|) \subsetneq \mathrm{Isom}(X, |||\cdot|||)$

transitive $\Rightarrow$ almost transitive $\Rightarrow$ maximal

Bounded subgroups and maximal norms

A subgroup G of $\mathrm{GL}(X, \|\cdot\|)$ is bounded if

$$\sup\{\|T\|: T \in G\} < \infty.$$

- If G is bounded then $G \subseteq \mathrm{Isom}(X, \|\!|\cdot|\!\|)$, where

$$\|\!|x|\!\| = \sup\{\|T(x)\|: T \in G\}$$

- G is a maximal bounded subgroup of $\mathrm{GL}(X) \Leftrightarrow (X, \|\!|\cdot|\!\|)$ is a maximal renorming of X and $G = \mathrm{Isom}(X, \|\!|\cdot|\!\|)$.

Question (Ferenczi-Rosendal, 2013)

Is every equivalent maximal (or almost transitive) norm on ℓ_2 euclidean?

Some Known Results

- (Banach) $L_p[0,1]$ with its usual norm is almost transitive.
- (Rolewicz) Non-hilbertian spaces with a 1-symmetric basis, e.g. ℓ_p, are maximal.
- (Lusky) Every separable Banach space is complemented in an almost transitive separable space.
- (Sanchez-Kawamura) $C[0,1]$ admits an almost transitive renorming.
- (Cabello) If X is almost transitive and has the *RNP* or is an Asplund space then X is super-reflexive.

Question (Deville-Godefroy-Zizler, 1993)

Does every super-reflexive space admit an almost transitive renorming?

Spaces without a maximal renorming

Question (Wood, 1982)

Does every X admit a maximal renorming?

i.e. does $\mathrm{GL}(X)$ have a maximal bounded subgroup?

Theorem (Ferenczi-Rosendal, 2013)

There exists a separable complex super-reflexive Banach space X (or a real reflexive space X) that does not admit any maximal renorming.

Corollary (Ferenczi-Rosendal)

There exists a separable complex super-reflexive Banach space X that does not admit any almost transitive renorming.

Almost transitive and maximal norms on ℓ_p

Question (Ferenczi-Rosendal)

Are there counterexamples among classical (in particular, non HI spaces)?

Theorem

ℓ_p (real or complex) does not admit an almost transitive renorming for $1 \leqslant p < \infty$ $(p \neq 2)$.

Question (Wood, 2006)

Is every bounded subgroup of $\mathrm{GL}(X)$ contained in a maximal bounded subgroup (assuming X has a maximal norm)?

Theorem

For $1 < p < \infty$ $(p \neq 2)$ there exist bounded subgroups of $\mathrm{GL}(\ell_p)$ that are not contained in any maximal subgroup.

Asymptotic Structure

Let X have a Schauder basis (e_i). A basis $(b_i)_{i=1}^n$ of unit vectors for an n-dimensional normed space belongs to the asymptotic structure $\{X,(e_i)\}_n$ if, $\forall\epsilon > 0, \forall m_1\,\exists x_1 > m_1$ s.t. $\forall m_2\,\exists x_2 > m_2$ s.t. $\dots\forall m_n\,\exists x_n > m_n$ s.t. for all scalars $(a_i)_{i=1}^n$,

$$\left\|\sum_{i=1}^n a_i b_i\right\| \leqslant \left\|\sum_{i=1}^n a_i x_i\right\| \leqslant (1+\varepsilon)\left\|\sum_{i=1}^n a_i b_i\right\|.$$

Theorem

Suppose X has an almost transitive subspace Y. Suppose ℓ_r is finitely representable in Y. Then, $\forall n \geqslant 2$ and $\forall a_1,\dots,a_{n-1},\exists(b_i)_{i=1}^n \in \{X,(e_i)\}_n$ s.t. $\forall$ scalars $\lambda, \forall 1 \leqslant k < n$,

$$\left\|\sum_{i=1}^k a_i b_i + \lambda b_{k+1}\right\| = \left(\sum_{i=1}^k |a_i|^r + |\lambda|^r\right)^{1/r}. \qquad (1)$$

In particular, the unit vector basis of ℓ_r^2 belongs to $\{X,(e_i)\}_2$.

Corollary

Suppose X has an almost transitive subspace Y. Then the unit vector basis of ℓ_2^2 belongs to $\{X, (e_i)\}_2$ and $\forall n \geqslant 3, \exists (b_i)_{i=1}^n \in \{X, (e_i)\}_n$ s.t.

$$\|\sum_{i=1}^{n} b_i\| = \sqrt{n}.$$

Proof.

By Dvoretzky's Theorem, ℓ_2 is finitely representable in Y. □

Applications

- Let X be an Asymptotic ℓ_p space $(1 < p < \infty, p \neq 2)$. Then no subspace of X admits an almost transitive norm.
- No subspace of a quotient space of ℓ_p $(1 < p < \infty, p \neq 2)$ admits an almost transitive norm.
- Let X be an Orlicz sequence space. Then X has a subspace admitting an almost transitive norm if and only if X has a subspace isomorphic to ℓ_2.
- Let Y be a subspace of $L_p[0,1]$ for $p > 2$. If Y admits an almost transitive norm then Y contains a subspace isomorphic to ℓ_2.
- If X is a stable Banach space and every subspace Y admits an almost transitive norm then X is ℓ_2-saturated.

Results on Bounded Subgroups

Two subgroups G, H of $\mathrm{GL}(X)$ are conjugate if $\exists T \in \mathrm{GL}(X)$ such that $H = T^{-1}GT$.

Theorem

ℓ_p ($1 < p < \infty, p \neq 2$) has a continuum of renormings whose isometry groups are pairwise non-conjugate and are not contained in any maximal bounded subgroup.

Remark

The proof generalizes to certain other spaces such as Pełczyński's space U with a universal unconditional basis and also the (weak Hilbert space) 2-convexified Tsirelson space $T^{(2)}$.

Results on Maximal Norms

Theorem

$GL(\ell_p)$ ($1 < p < \infty, p \neq 2$) has a continuum of maximal bounded subgroups that are pairwise non-conjugate.

Theorem

Let X be a non-hilbertian space with a symmetric basis. Then $\mathrm{GL}(X)$ *has infinitely many non-conjugate maximal bounded subgroups.*

Question (Ferenczi-Rosendal)

Is every equivalent maximal norm on ℓ_2 euclidean?

Functional Hilbertian Sums

A normalized 1-unconditional basis (e_i) is either impure (definition below), e.g. the u.v.b. of ℓ_2

or pure, e.g. the u.v.b. of ℓ_p for $p \neq 2$.

(e_i) is called impure if there exist $i < j$ so that (e_i, e_j) is isometrically equivalent to the u.v.b. of ℓ_2^2 and for all $x, x' \in \text{span}(e_i, e_j)$ with $\|x\| = \|x'\|$ and for all $y \in \text{span}\{e_k : k \neq i, j\}$ we have $\|x + y\| = \|x' + y\|$.

Theorem (H. Rosenthal, 1988)

Let $E = (e_i)$ be a pure 1-unconditional basis and (H_i) be Hilbert spaces all of dimension at least 2, and let $Z = (\sum \oplus H_i)_E$ be the corresponding functional hilbertian sum:

$$\|(h_i)\|_Z = \left\| \sum \|h_i\| e_i \right\|_E \qquad (h_i \in H_i).$$

The surjective isometries of Z are just the "obvious" ones which permute E and send fibres to fibres via Hilbert space isometries.

Theorem

Let $E = \{e_k\}_{k=1}^{\infty}$ *be a pure non-Hilbertian* 1*-symmetric basis and let* $Z = (\sum_{k=1}^{\infty} \oplus \ell_2^k)_E$*.* $\exists$ *a continuum of renormings of* Z *whose isometry groups are not contained in any maximal bounded subgroup and are not pairwise conjugate.*

Let J *be any infinite subset of* $\mathbb{N}$*. For* $j \in J$*, let* $H_j = \ell_2^{2^j}$*, and let*

$$Z_J = Z_J(E) = \left(\sum_{j \in J} \oplus H_j \right)_E .$$

Then Z_J *is isomorphic to* Z*,* $\mathrm{Isom}(Z_J)$ *is not contained in any maximal bounded subgroup of* $\mathrm{GL}(Z)$*, and, if* $J \neq J'$ *then* $\mathrm{Isom}(Z_J)$ *and* $\mathrm{Isom}(Z_{J'})$ *are not conjugate in* $\mathrm{GL}(Z)$

Corollary

For $1 < p < \infty$, $p \neq 2$, $\exists$ a continuum of renormings of ℓ_p whose isometry groups are not contained in any maximal bounded subgroup and are not pairwise conjugate.

Proof.

Take E to be the u.v.b. of ℓ_p. Then $Z = (\sum_{k=1}^{\infty} \oplus \ell_2^k)_E$ is isomorphic to ℓ_p since ℓ_p is prime (Pełczyński, 1964). □

Corollary

The result holds for Pełczyński's space U with a universal unconditional basis.

Proof.

U has a symmetric basis E such that $Z = (\sum_{k=1}^{\infty} \oplus \ell_2^k)_E$ is isomorphic to U which follows from the universality property of U and the Pełczyński decomposition argument. □

Remark

The result also holds for 2-convexified Tsirelson space $T^{(2)}$.

Theorem

Let $E = \{e_k\}_{k=1}^{\infty}$ *be a pure* 1*-symmetric basis and let* $Z = (\sum_{k=1}^{\infty} \oplus \ell_2^k)_E$. $\exists$ *a continuum of renormings of* Z *whose isometry groups are maximal and not pairwise conjugate in the isomorphism group of* Z.

Corollary

For $1 < p < \infty$, $p \neq 2$, $\exists$ *a continuum of renormings of* ℓ_p *whose isometry groups are maximal and not pairwise conjugate in the isomorphism group of* ℓ_p.

Remark

The same result holds for U.