Hadamard type operators for real analytic functions of several variables and moments of analytic functionals

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(based on joint results with Michael Langenbruch – Oldenburg)

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Difference

translations: a group dilations: a semigroup

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Definition

$f_n \to f$ in $\mathscr{A}(\mathbb{R}^d)$ iff $\exists$ U a complex neighbourhood of $\mathbb{R}^d$ s.t. $f_n, f \in H(U)$ and $f_n \to f$ in $H(U)$.

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Operators as above are called (Hadamard) multipliers on $\mathscr{A}(\mathbb{R}^d)$.

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- Which multipliers can be inverted or “partially inverted”?

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For $d = 1$: Arakelyan 1980, for arbitrary d: D-Langenbruch 2014

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Observation

im $P(\theta) \supset \mathscr{A}_+(\mathbb{R}^d)$ iff $\left(\frac{1}{P(\alpha)}\right)_{\alpha>0}$ is a multiplier sequence.

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Theorem (Fiedler-Gregor 1981)

A quadratic form P has the Hurwitz property iff it is proportional to a quadratic form P_1 with non-negative real coefficients and with the positive signature $n_+(P_1) = 1$.

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- $\theta_1^2 - \theta_2^2 - \theta_3^2 - \cdots - \theta_k^2$, $k > 1$: NO "wave-Euler";

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Example (im $P(\theta) \supset \mathscr{A}_+(\mathbb{R}^d)$?)

- $\theta_1^2 + \cdots + \theta_k^2$, $k > 1$: NO "Laplace-Euler";
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- $\theta_1^2 + \theta_2^2 + \cdots + \theta_k^2 - \theta_{k+1}$, $k > 1$: NO "heat-Euler";

8. Examples of surjective Euler operators

Theorem

For any homogeneous polynomial P of d variables TFAE

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Example (Surjective operator on $\mathscr{A}(\mathbb{R}^d)$)

$$\theta_1\theta_2 + \theta_2\theta_3 + \theta_1\theta_3 + 2(\theta_1 + \theta_2 + \theta_3) + 3$$