Hadamard type operators for real analytic functions of several variables and moments of analytic functionals

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(based on joint results with Michael Langenbruch – Oldenburg)

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Pełczyński Memorial Conference
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**Definition (dilation)**

$$M_a(f)(x) := f(ax) \quad ax = (a_1x_1, \ldots, a_dx_d) \quad a, x \in \mathbb{R}^d$$
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**Difference**

Translations: a group  
Dilations: a semigroup
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Notation

$\mathcal{A}(\mathbb{R}^d)$ — the class of real analytic functions
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Topology: nice, complete, nuclear, closed graph theorem, uniform boundedness principle
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A linear operator on \( \mathcal{A}(\mathbb{R}^d) \) is continuous iff it is sequentially continuous.
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A linear operator on \( \mathscr{A}(\mathbb{R}^d) \) is continuous iff it is sequentially continuous.

**Definition**

\[ f_n \to f \text{ in } \mathscr{A}(\mathbb{R}^d) \text{ iff } \exists U \text{ a complex neighbourhood of } \mathbb{R}^d \text{ s.t. } f_n, f \in H(U) \text{ and } f_n \to f \text{ in } H(U). \]
3. Hadamard multipliers

Theorem
Let $L : A(\mathbb{R}^d) \rightarrow A(\mathbb{R}^d)$ be a linear continuous map. TFAE

(a) $L M_a = M_a L \quad \forall a \in \mathbb{R}^d$;
(b) (monomials are eigenvectors) $\forall \alpha \in \mathbb{N}^d \ L x^\alpha = m_\alpha x^\alpha$, $m_\alpha \in \mathbb{C}$ — the multiplier sequence;
(c) (multiplicative convolution) $\exists! T \in A(\mathbb{R}^d)' \ L (f)(x) = \langle T, M_x(f) \rangle = \langle T, f(x \cdot) \rangle$ — the moment sequence.

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Operators as above are called (Hadamard) multipliers on $A(\mathbb{R}^d)$. 
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Let \( L : \mathcal{A}(\mathbb{R}^d) \to \mathcal{A}(\mathbb{R}^d) \) be a linear continuous map. TFAE

- (a)
  \( LM_a = M_a L \) for all \( a \in \mathbb{R}^d \);
  \( M_a(f)(x) = f(ax) \)

- (b) (monomials are eigenvectors)
  \( \forall \alpha \in \mathbb{N}^d, L x^\alpha = m_\alpha x^\alpha \)
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- (c) (multiplicative convolution)
  \( \exists ! T \in \mathcal{A}(\mathbb{R}^d)' \)
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4. Examples

Example (Euler partial differential operators)

$$P(z) = \sum_{|\alpha| \leq q} a_\alpha z^{\alpha}, z \in \mathbb{C}^d$$ — a polynomial

$$\theta_j(f) = x_j \frac{\partial f}{\partial x_j}$$

$$P(\theta) := \sum_{|\alpha| \leq q} a_\alpha \theta^{\alpha}$$

$$= \sum_{|\alpha_1| \leq q} a_{1\alpha_1} \cdots \theta_{d \alpha_d}$$ — an Euler pdo

$$\alpha \in \mathbb{N}^d$$ — the multiplier sequence

Problems

Describe multiplier sequences = moment sequences for analytic functionals.

Which multipliers can be inverted or "partially inverted"?
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**Definition (Class of holomorphic functions)**

\[ H := \{ f \in H(C^d) | f(\beta) = 0 \forall \beta \in \mathbb{Z}^d \setminus \mathbb{N}^d \text{ and } \forall n \in \mathbb{N} \sup_{z \in \omega_d^n} |f(z)| \exp(-n \sum_{j=1}^{d} Re z_j) < \infty \} \]
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\[ a \in \mathbb{R}, \ v_n \searrow -\infty, \ k_n \nearrow +\infty, \ \omega_n = v_n + \{ z = x + iy \in \mathbb{C} : |y| < k_nx \} \].

Theorem (Interpolation theorem for multiplier sequences)

The map \( F^+ : H_a \to A([0, e^a]^d)', \langle F^+ (f), x^{\alpha} \rangle = f(\alpha), \ \alpha \in \mathbb{N}^d \) is a well-defined linear continuous surjective map, where \( f \in \ker F^+ \) iff:

\[
\sum_{d,k=1}^{k_n} \sin(\pi z_k) g_k(z) = 0,
\]
where

\[
\sup_{z \in \omega_d} |g_k(z)| \exp\left( -\left( a + 1 \right) \sum_{j \neq k} \Re z_j + n \Re z_k \right) < \infty.
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For \( d = 1 \): Arakelyan 1980, for arbitrary \( d \): D-Langenbruch 2014.
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\[ \forall \ n \in \mathbb{N} \ \sup_{z \in \omega_n^d} |f(z)| \exp \left( - \left( a + \frac{1}{n} \right) \sum_j \text{Re } z_j \right) < \infty \} \]
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\( T \in \mathcal{A}(\mathbb{R}^d)' : \text{supp} \ T := \min. \text{ cpct } K \text{ s.t. } T \in H(U)' \ \forall \ U \text{ nbhd of } K \)
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**Theorem (Interpolation theorem for multiplier sequences)**

*The map* \( F_+ : \mathcal{H}_a \rightarrow \mathcal{A}([0, e^a]^d)', \ \langle F_+(f), x^\alpha \rangle = f(\alpha), \ \alpha \in \mathbb{N}^d, \) *is a well-defined linear continuous surjective map*
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*The map \( F_+ : \mathcal{H}_a \rightarrow \mathcal{A}([0, e^a]^d)' \), \( \langle F_+(f), x^\alpha \rangle = f(\alpha), \; \alpha \in \mathbb{N}^d \), is a well-defined linear continuous surjective map, where \( f \in \ker F_+ \) iff: \( f(z) = \sum_{k=1}^d \sin(\pi z_k)g_k(z) \)
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$$\mathcal{A}_+(\mathbb{R}^d) := \{ f \in \mathcal{A}(\mathbb{R}^d) : f = \sum_{\alpha > 0} f_\alpha z^\alpha \}$$
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(b) $P$ has the Hurwitz property,
6. Surjectivity of Euler operators $P(\theta)$

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$$\mathcal{A}_+(\mathbb{R}^d) := \{ f \in \mathcal{A}(\mathbb{R}^d) : f = \sum_{\alpha > 0} f_\alpha z^\alpha \}$$

Theorem

For any \textit{homogeneous} polynomial $P$ of $d$ variables TFAE

(a) $\text{im } P(\theta) \supset \mathcal{A}_+(\mathbb{R}^d)$;

(b) $P$ has the Hurwitz property, i.e., it has no zeros in $\mathbb{C}_+^d$: the product of the right halfplanes.
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**Observation**

\( \text{im } P(\theta) \supset \mathcal{A}_+(\mathbb{R}^d) \) iff \( \left( \frac{1}{P(\alpha)} \right)_{\alpha > 0} \) is a multiplier sequence.
7. Polynomials with the Hurwitz property

Theorem (Choe-Oxley-Sokal-Wagner 2004)
A homogeneous polynomial with the Hurwitz property is proportional to a polynomial with all coefficients real non-negative.

Theorem (Fettweis 1990)
Every elementary symmetric polynomial has the Hurwitz property.

Theorem (Fiedler-Gregor 1981)
A quadratic form $P$ has the Hurwitz property iff it is proportional to a quadratic form $P_1$ with non-negative real coefficients and with the positive signature $n_+ (P_1) = 1$. 
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**Example (im $P(\theta) \supset \mathcal{A}_+(\mathbb{R}^d)$?)**
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Example (im $P(\theta) \supset \mathcal{A}_+(\mathbb{R}^d)$?)

- $\theta_1^2 + \cdots + \theta_k^2, k > 1$: NO  
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8. Examples of surjective Euler operators

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**Example** (im $P(\theta) \supset \mathcal{A}_+ (\mathbb{R}^d)$?)

- $\theta_1^2 + \cdots + \theta_k^2, \ k > 1$: NO “Laplace-Euler”;
- $\theta_1^2 - \theta_2^2 - \theta_3^2 - \cdots - \theta_k^2, \ k > 1$: NO “wave-Euler”;
- $\theta_k^2 + \cdots + \theta_2^2 + \theta_1^2$: NO
8. Examples of surjective Euler operators

**Theorem**

For any homogeneous polynomial $P$ of $d$ variables TFAE

(a) $\text{im } P(\theta) \supset A_+(\mathbb{R}^d)$;

(b) $P$ has the Hurwitz property, i.e., it has no zeros in $\mathbb{C}_+^d$.

$A_+(\mathbb{R}^d) := \{ f \in A(\mathbb{R}^d) : f = \sum_{\alpha_1>0, \alpha_2>0, \ldots, \alpha_d>0} f_\alpha z^\alpha \}$

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  “Laplace-Euler”;

- $\theta_1^2 - \theta_2^2 - \theta_3^2 - \cdots - \theta_k^2, \ k > 1$: NO
  
  “wave-Euler”;

- $\theta_1^2 + \theta_2^2 + \cdots + \theta_k^2 - \theta_{k+1}, \ k > 1$: NO
  
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For any homogeneous polynomial $P$ of $d$ variables TFAE

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- $\theta_1^2 + \cdots + \theta_k^2, k > 1$: NO  
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  “wave-Euler”;

- $\theta_1^2 + \theta_2^2 + \cdots + \theta_k^2 - \theta_{k+1}, k > 1$: NO  
  “heat-Euler”;

- $\theta_1 \theta_2 + \theta_2 \theta_3 + \theta_1 \theta_3$: YES;
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For any homogeneous polynomial $P$ of $d$ variables TFAE

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- $\theta_1^2 + \cdots + \theta_k^2, \ k > 1$: NO  "Laplace-Euler";
- $\theta_1^2 - \theta_2^2 - \theta_3^2 - \cdots - \theta_k^2, \ k > 1$: NO  "wave-Euler";
- $\theta_1^2 + \theta_2^2 + \cdots + \theta_k^2 - \theta_{k+1}, \ k > 1$: NO  "heat-Euler";
- $\theta_1\theta_2 + \theta_2\theta_3 + \theta_1\theta_3$: YES;

Example (Surjective operator on $\mathcal{A}(\mathbb{R}^d)$)

$$\theta_1\theta_2 + \theta_2\theta_3 + \theta_1\theta_3 + 2(\theta_1 + \theta_2 + \theta_3) + 3$$