Mean-width and mean-norm of isotropic convex bodies

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M and M^*

- We assume that K is a centrally symmetric convex body of volume 1 in $\mathbb{R}^n$:

$$K = \{x \in \mathbb{R}^n : \|x\| \leqslant 1\}.$$

- The mean-norm of K is defined by

$$M(K) = \int_{S^{n-1}} \|x\| \, d\sigma(x).$$

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$$M(K) = \int_{S^{n-1}} \|x\| \, d\sigma(x).$$

- The support function of K is

$$h_K(x) = \|x\|_* = \max\{\langle x, y\rangle : y \in K\},$$

and the mean-width of K is

$$M^*(K) = w(K) = \int_{S^{n-1}} h_K(x) \, d\sigma(x).$$

Lower bounds

- Using integration in polar coordinates and Hölder's inequality we get

$$M(K) \geqslant \left(\frac{|B_2^n|}{|K|}\right)^{1/n} \geqslant \frac{c_1}{\sqrt{n}}.$$

- From Urysohn's inequality,

$$M^*(K) \geqslant \mathrm{vrad}(K) := \left(\frac{|K|}{|B_2^n|}\right)^{1/n} \geqslant c_2\sqrt{n}.$$

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- These lower bounds for M and M^* are sharp: if $D_n = r_n B_2^n$ has volume 1 then $r_n \simeq \sqrt{n}$ and

$$M(D_n) = \frac{1}{r_n} \simeq \frac{1}{\sqrt{n}} \quad \textit{while} \quad M^*(D_n) = r_n \simeq \sqrt{n}.$$

Upper bounds

Theorem (Lewis, Figiel-Tomczak, Pisier)

Every centrally symmetric convex body K in $\mathbb{R}^n$ has a linear image (a position) $\tilde{K}$ of volume 1 such that

$$M(\tilde{K})\, M^*(\tilde{K}) \leqslant c_1 \log[d(X_K, \ell_2^n) + 1] \leqslant c_2 \log n.$$

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$$M(\tilde{K})\, M^*(\tilde{K}) \leqslant c_1 \log[d(X_K, \ell_2^n) + 1] \leqslant c_2 \log n.$$

- For this position of K, using the previous lower bounds, we have

$$M(\tilde{K}) \leqslant \frac{c \log n}{\sqrt{n}} \quad \text{and} \quad M^*(\tilde{K}) \leqslant c\sqrt{n} \log n.$$

- **Question:** What can we say about the isotropic position?

Isotropic convex bodies

Isotropic convex bodies

A convex body K in $\mathbb{R}^n$ is called isotropic if it has volume 1, it is centered, and there exists a constant $L_K > 0$ such that

$$\int_K \langle x, \theta \rangle^2 dx = L_K^2$$

for every $\theta \in S^{n-1}$.

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Hyperplane conjecture

There exists an absolute constant $C > 0$ such that $L_K \leqslant C$ for every n and every isotropic convex body K in $\mathbb{R}^n$.
Bourgain: $L_K \leqslant c\sqrt[4]{n}\log n$, Klartag: $L_K \leqslant c\sqrt[4]{n}$.

Log-concave measures

Log-concave measures

A measure μ on $\mathbb{R}^n$ is called log-concave if

$$\mu(\lambda A + (1-\lambda)B) \geqslant \mu(A)^\lambda \mu(B)^{1-\lambda}$$

for any non-empty compact subsets A and B of $\mathbb{R}^n$ and any $\lambda \in (0,1)$.

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Isotropic log-concave measures

We say that a log-concave probability measure μ is isotropic if $\mathrm{bar}(\mu) = 0$ and $\mathrm{Cov}(\mu)$ is the identity matrix:

$$\int x_i x_j f_\mu(x)dx = \delta_{ij}.$$

Then, the isotropic constant of μ is $L_\mu = \|f_\mu\|_\infty^{1/n} \simeq f_\mu(0)^{1/n}$.

Isotropic log-concave measures

If K is a convex body in $\mathbb{R}^n$, then the Brunn-Minkowski inequality implies that $\mathbf{1}_K$ is the density of a log-concave measure. K is isotropic if and only if the measure μ_K with density $L_K^n \mathbf{1}_{\frac{1}{L_K}K}$ is isotropic.

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Marginal

The marginal of μ with respect to $F \in G_{n,k}$ is defined by

$$\pi_F \mu(A) := \mu(P_F^{-1}(A)) = \mu(A + F^{\perp})$$

for all Borel subsets of F. The density of $\pi_F \mu$ is the function

$$f_{\pi_F \mu}(x) = \int_{x+F^{\perp}} f_\mu(y) dy, \qquad x \in F.$$

If μ is centered, log-concave or isotropic, then $\pi_F \mu$ is respectively also centered, log-concave or isotropic.

L_q-centroid bodies

L_q-centroid bodies

If μ is a probability measure on $\mathbb{R}^n$, the L_q-centroid body $Z_q(\mu)$, $q \geqslant 1$, is the symmetric convex body with support function

$$h_{Z_q(\mu)}(y) := \|\langle \cdot, y\rangle\|_{L^q(\mu)} = \left(\int |\langle x, y\rangle|^q d\mu(x)\right)^{1/q}.$$

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- μ is isotropic if and only if it is centered and $Z_2(\mu) = B_2^n$.
- From Hölder's inequality it follows that $Z_2(\mu) \subseteq Z_p(\mu) \subseteq Z_q(\mu)$ for all $2 \leqslant p \leqslant q < \infty$.
- From Borell's lemma, $Z_q(\mu) \subseteq c\frac{q}{p} Z_p(\mu)$ for all $2 \leqslant p < q$.
- If μ is isotropic, then $R(Z_q(\mu)) := \max\{h_{Z_q(\mu)}(\theta) : \theta \in S^{n-1}\} \leqslant cq$.

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If K is a convex body of volume 1 in $\mathbb{R}^n$, the L_q-centroid body $Z_q(K)$, $q \geqslant 1$, is the symmetric convex body with support function

$$h_{Z_q(K)}(y) := \left(\int_K |\langle x, y \rangle|^q dx \right)^{1/q}.$$

- K is isotropic if and only if it is centered and $Z_2(K) = L_K B_2^n$.
- If K is centrally symmetric then

$$cK \subseteq Z_q(K) \subseteq K$$

  for all $q \geqslant n$.
- If K is isotropic and if μ_K is the isotropic measure with density $L_K^n \mathbf{1}_{\frac{1}{L_K}K}$, then

$$Z_q(K) = L_K \, Z_q(\mu_K).$$

The two questions

Assume that K is centrally symmetric and isotropic in $\mathbb{R}^n$.

Question 1

To give an upper bound for $M^*(K)$.

From the inclusion $K \subseteq (n+1)L_K\, B_2^n$, one has the obvious bound $M^*(K) \leqslant (n+1)L_K$.
Until recently, it was known that $M^*(K) \leqslant cn^{3/4} L_K$.
Several approaches: Hartzoulaki, Pivovarov, "Z_q-bound".

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Question 2

To give an upper bound for $M(K)$.

From the inclusion $K \supseteq L_K\, B_2^n$, one has the obvious bound $M(K) \leqslant 1/L_K$.
Until recently, there was no lower bound depending on n.

A first idea: Dudley's entropy estimate

Entropy numbers

The covering number $N(K, T)$ of K by T is the minimal number of translates of T whose union covers K. For any $k \geqslant 1$ we set

$$e_k(K, T) := \inf\{s > 0 : N(K, sT) \leqslant 2^k\}.$$

The k-th entropy number of K is $e_k(K) := e_k(K, B_2^n)$.

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The k-th entropy number of K is $e_k(K) := e_k(K, B_2^n)$.

Dudley's bound

If K is a centrally symmetric convex body in $\mathbb{R}^n$ then

$$\sqrt{n}M^*(K) \leqslant c_1 \sum_{k \geqslant 1} \frac{1}{\sqrt{k}} e_k(K, B_2^n).$$

A first idea: Dudley's entropy estimate

Covering numbers

If K is an isotropic convex body in $\mathbb{R}^n$ then

$$\log N(K, sB_2^n) \leqslant C_1 \frac{n^{3/2} L_K}{s}$$

for all $s > 0$. Therefore,

$$e_k(K, B_2^n) = \inf\{s > 0 : N(K, sB_2^n) \leqslant 2^k\} \leqslant C_2 \sqrt{n} L_K \frac{n}{k}.$$

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Then, we combine this with Dudley's bound

$$M^*(K) \leqslant c_1 \sum_{k \geqslant 1} \frac{1}{\sqrt{k}} e_k(K, B_2^n)$$

to get $M^*(K) \leqslant C n^{3/4} L_K$.

A refinement by V. Milman and Pisier

The parameter $v_k(K)$

For any $k \geqslant 1$ we set

$$v_k(K) := \sup\{\mathrm{vrad}(P_F(K)) : F \in G_{n,k}\}.$$

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v_k and e_k

For every $F \in G_{n,k}$ we have

$$\begin{aligned} |P_F(K)| &\leqslant N(P_F(K), e_k P_F(B_2^n))|e_k B_F| \\ &\leqslant N(K, e_k(K) B_2^n) e_k^k |B_F| \leqslant (2e_k)^k |B_F|, \end{aligned}$$

therefore

$$v_k(K) \leqslant 2e_k(K).$$

A refinement by V. Milman and Pisier

Theorem (V. Milman-Pisier)

For every centrally symmetric convex body K in $\mathbb{R}^n$ one has

$$\sqrt{n}M^*(K) \leqslant c_2 \sum_{k=1}^{n} \frac{1}{\sqrt{k}} \mathrm{Rad}_k(K) v_k(K).$$

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$\mathrm{Rad}_k(K)$

In the statement above, $\mathrm{Rad}_k(K) := \sup\{\mathrm{Rad}(X_{P_F(K)}) : F \in G_{n,k}\}$ where $\mathrm{Rad}(Y) \leqslant c_3 \log(d(Y, \ell_2^{\dim(Y)}) + 1)$ is the Rademacher constant of Y.

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- So, roughly speaking,

$$\sqrt{n}M^*(K) \lesssim \sum_{k=1}^{n} \frac{1}{\sqrt{k}} v_k(K).$$

Mean width: isotropic case

- We know that if K is a centrally symmetric isotropic convex body in $\mathbb{R}^n$ then

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- So, we look for an upper bound for $M^*(Z_n(K))$, and more generally for $M^*(Z_q(K))$. To this end we use

$$\sqrt{n} M^*(Z_q(K)) \lesssim \sum_{k=1}^{n} \frac{1}{\sqrt{k}} v_k(Z_q(K)).$$

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- So, we look for an upper bound for $M^*(Z_n(K))$, and more generally for $M^*(Z_q(K))$. To this end we use
$$\sqrt{n}M^*(Z_q(K)) \lesssim \sum_{k=1}^{n} \frac{1}{\sqrt{k}} v_k(Z_q(K)).$$
- In order to estimate $v_k(Z_q(K))$ we consider any $F \in G_{n,k}$ and try to give an upper bound for
$$\mathrm{vrad}(P_F(Z_q(K))) = \left(\frac{|P_F(Z_q(K))|}{|B_2^k|}\right)^{1/k}.$$

Mean width: bound for $v_k(Z_q(K))$

Projections of $P_F(\mu)$

For any $F \in G_{n,k}$ we have $P_F(Z_q(\mu)) = Z_q(\pi_F(\mu))$.

Mean width: bound for $v_k(Z_q(K))$

Projections of $P_F(\mu)$

For any $F \in G_{n,k}$ we have $P_F(Z_q(\mu)) = Z_q(\pi_F(\mu))$.

Since $\pi_F(\mu)$ is an isotropic log-concave measure on F we may use:

Theorem (Paouris)

If ν is an isotropic log-concave measure on $\mathbb{R}^k$ then

$$\mathrm{vrad}(Z_q(\nu)) \leqslant c_5\sqrt{q} \quad \textit{if } q \leqslant k$$

and

$$\mathrm{vrad}(Z_q(\nu)) \leqslant c_5(q/k)\sqrt{k} \quad \textit{if } q \geqslant k$$

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- Applied to $\mu = \mu_K$, this gives

$$v_k(Z_q(K)) \leqslant c_6\sqrt{\frac{q}{k}}\max(\sqrt{q}, \sqrt{k})L_K.$$

Mean width: E. Milman's bound

Theorem (E. Milman)

For every centrally symmetric isotropic convex body K in $\mathbb{R}^n$ and for every $2 \leqslant q \leqslant n$,

$$M^*(Z_q(K)) \leqslant c\sqrt{q}(\log q)^2 L_K.$$

In particular,

$$M^*(K) \leqslant c\sqrt{n}(\log n)^2 L_K.$$

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For the proof we write

$$L_K^{-1}\sqrt{n}M^*(Z_q(K)) \lesssim L_K^{-1}\sum_{k=1}^{n}\frac{1}{\sqrt{k}}v_k(z_q(K)) \leqslant c_2\sum_{k=1}^{n}\max\left(\sqrt{\frac{q}{k}}, \frac{q}{k}\right)$$
$$\simeq q\sum_{k=1}^{q}\frac{1}{k} + \sqrt{q}\sum_{k=q}^{n}\frac{1}{\sqrt{q}} \simeq q\log q + \sqrt{q}\sqrt{n} \leqslant \sqrt{n}\sqrt{q}\log q.$$

Mean-norm: the dual problem

- Let K be a centrally symmetric isotropic convex body K in $\mathbb{R}^n$. In order to estimate $M(K)$ we may start from the estimate of V. Milman and Pisier, using duality: we have

$$\sqrt{n}M(K) \leqslant c_2 \sum_{k=1}^{n} \frac{1}{\sqrt{k}} \mathrm{Rad}_k(K^\circ) v_k(K^\circ).$$

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- Note that

$$\begin{aligned} v_k(K^\circ) &:= \sup\{\mathrm{vrad}(P_F(K^\circ)) : F \in G_{n,k}\} \\ &\simeq \frac{1}{\inf\{\mathrm{vrad}(K \cap F) : F \in G_{n,k}\}} =: \frac{1}{w_k^-(K)}. \end{aligned}$$

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- Because of the formula $P_F(Z_q(\mu)) = Z_q(\pi_F(\mu))$ we would prefer to work with the quantity

$$v_k^-(K) := \inf\{\mathrm{vrad}(P_F(K)) : F \in G_{n,k}\}.$$

Note that $w_k^-(K) \leqslant v_k^-(K)$.

Mean norm: new general bound

Theorem (G.-E. Milman)

For every centrally symmetric convex body K in $\mathbb{R}^n$ and $k \geqslant 1$ we have

$$e_k(K^\circ, B_2^n) \leqslant C\frac{n}{k}\log\left(e+\frac{n}{k}\right)\sup_{1\leqslant m\leqslant \min(k,n)}\frac{1}{2^{\frac{k}{3m}}v_m^-(K)}.$$

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This leads to the next general bound:

Theorem (G.-E. Milman)

Let K be a centrally symmetric convex body in $\mathbb{R}^n$ with $K \supseteq rB_2^n$. Then,

$$\sqrt{n}M(K) \leqslant C\sum_{k=1}^{n}\frac{1}{\sqrt{k}}\min\left(\frac{1}{r},\frac{n}{k}\log\left(e+\frac{n}{k}\right)\frac{1}{v_k^-(K)}\right).$$

Mean norm of $Z_q(\mu)$

- We need a lower bound for $\mathrm{vrad}(P_F(Z_q(\mu))) = \mathrm{vrad}(Z_q(\pi_F(\mu)))$, when $F \in G_{n,k}$.

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- We need a lower bound for $\mathrm{vrad}(P_F(Z_q(\mu))) = \mathrm{vrad}(Z_q(\pi_F(\mu)))$, when $F \in G_{n,k}$.
- The main tool is a theorem of Klartag and E. Milman: if ν is an isotropic log-concave measure on $\mathbb{R}^k$ then, for all $1 \leqslant q \leqslant \sqrt{k}$,

$$\mathrm{vrad}(Z_q(\nu)) \geqslant c\sqrt{q}.$$

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- It follows that if μ is an isotropic log-concave measure on $\mathbb{R}^n$ then, for any $q \geqslant 1$ and $k = 1, \ldots, n$ we have

$$v_k^-(Z_q(\mu)) \geqslant c\sqrt{\min(q, \sqrt{k})}.$$

Mean norm of $Z_q(\mu)$

Theorem (G.-E. Milman)

For any isotropic log-concave measure μ on $\mathbb{R}^n$ and any $q \leqslant q_0 := (n \log(e+n))^{2/5}$,

$$M(Z_q(\mu)) \leqslant C \frac{\sqrt{\log q}}{\sqrt[4]{q}}.$$

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$$M(Z_q(\mu)) \leqslant C \frac{\sqrt{\log q}}{\sqrt[4]{q}}.$$

Since $M(K) \leqslant M(Z_{q_0}(K))$ we have:

Theorem (G.-E. Milman)

For any centrally symmetric isotropic convex body K in $\mathbb{R}^n$ we have

$$M(K) \leqslant \frac{C}{L_K} \frac{\log^{2/5}(e+n)}{n^{1/10}}.$$

One more problem

- Paouris has proved that if μ is an isotropic log-concave measure on $\mathbb{R}^n$, then
$$L_\mu \simeq \frac{C\sqrt{n}}{\mathrm{vrad}(Z_n(\mu))} \leqslant \frac{C\sqrt{n}}{\mathrm{vrad}(Z_q(\mu))}$$
for all $q \leqslant n$.
- Klartag and E. Milman have proved that $\mathrm{vrad}(Z_{\sqrt{n}}(\mu)) \geqslant c\sqrt[4]{n}$.
- This gives a proof of the best known bound $L_\mu \leqslant C\sqrt[4]{n}$.

Question

To prove that $M^*(Z_q(\mu)) \geqslant c\sqrt{q}$ for $q \gg \sqrt{n}$.

- By Urysohn's inequality this is "less" than showing that $\mathrm{vrad}(Z_q(\mu)) \geqslant c\sqrt{q}$ for $q \gg \sqrt{n}$.

Applications: I. Sub-Gaussian directions

- A direction $\theta \in S^{n-1}$ is a ψ_α-direction (where $1 \leqslant \alpha \leqslant 2$) for K with constant $b > 0$ if
$$\|\langle \cdot, \theta \rangle\|_{\psi_\alpha} \leqslant b \|\langle \cdot, \theta \rangle\|_2,$$
where
$$\|\langle \cdot, \theta \rangle\|_{\psi_\alpha} := \inf \left\{ t > 0 : \int_K \exp\left(\frac{|\langle x, \theta \rangle|}{t} \right)^\alpha dx \leqslant 2 \right\}.$$

- It is known that
$$\|\langle \cdot, \theta \rangle\|_{\psi_\alpha} \simeq \sup_{q \geqslant \alpha} \frac{\|\langle \cdot, \theta \rangle\|_q}{q^{1/\alpha}}.$$

Applications: I. Sub-Gaussian directions

- From the Brunn-Minkowski inequality it follows that every $\theta \in S^{n-1}$ is a ψ_1-direction for K with an absolute constant C.
- Question: is it true that there exists an absolute constant $C > 0$ such that every K has at least one sub-Gaussian direction (ψ_2-direction) with constant C?
- It is known that the answer is affirmative, with a constant $O(\sqrt{\log n})$. This is due to Paouris, Valettas and G. (2011). The first result of this type was proved by Klartag (2006).
- In the isotropic case: is it true that most $\theta \in S^{n-1}$ are sub-Gaussian directions for K with a constant at most logarithmic in n?

Applications: I. Sub-Gaussian directions

Theorem (Brazitikos-Hioni)

Let K be an isotropic symmetric convex body in $\mathbb{R}^n$. Then,

$$\int_{S^{n-1}} \|\langle \cdot, \theta\rangle\|_{\psi_2} d\sigma(\theta) \leqslant C(\log n)^3 L_K,$$

where $C > 0$ is an absolute constant.

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Theorem (Brazitikos-Hioni)

Let K be an isotropic symmetric convex body in $\mathbb{R}^n$. Then, for any $\alpha > 1$ we have

$$\|\langle\cdot,\theta\rangle\|_{\psi_2} \leqslant C\sqrt{\alpha}(\log n)^{3/2} \max\left\{\frac{\sqrt{\log n}L_K}{\sqrt{\alpha}}, L_K^2\right\}$$

for all θ in a subset Θ of S^{n-1} with $\sigma(\Theta) \geqslant 1 - n^{-\alpha}$, where $C > 0$ is an absolute constant.

Applications: I. Sub-Gaussian directions

- Proof of the first theorem: for any $\theta \in S^{n-1}$ we have

$$\|\langle \cdot, \theta\rangle\|_{\psi_2} \leqslant C_1 \max_{1\leqslant s\leqslant m} \frac{h_{Z_{2^s}(K)}(y)}{2^{s/2}}$$

where $m = \lfloor \log_2 n \rfloor$. It trivially follows that

$$\mathbb{E}_\theta(\|\langle \cdot, \theta\rangle\|_{\psi_2}) \leqslant C_1 \sum_{s=1}^{m} \frac{M^*(Z_{2^s}(K))}{2^{s/2}}.$$

- We know that

$$M^*(Z_{2^s}(K)) \leqslant C_2 s 2^{s/2} \max\left\{\frac{s2^{s/2}}{\sqrt{n}}, 1\right\} L_K.$$

- Therefore,

$$\mathbb{E}_\theta(\|\langle \cdot, \theta\rangle\|_{\psi_2}) \leqslant C_3 \sum_{s=1}^{m} s \max\left\{\frac{s2^{s/2}}{\sqrt{n}}, 1\right\} L_K$$

$$\leqslant C_3 m^3 L_K \leqslant C_4 (\log n)^3 L_K.$$

Applications: II. Rogers-Shephard inequality

- Let K be a convex body of volume 1 in $\mathbb{R}^n$ with $0 \in \mathrm{int}(K)$. For every $1 \leqslant k \leqslant n-1$ and any $F \in G_{n,k}$ we define

$$g(K,k;F) := \left(|P_F(K)|\,|K \cap F^\perp|\right)^{1/k},$$

where $F^\perp$ denotes the orthogonal subspace of F in $\mathbb{R}^n$.

- A classical inequality of Rogers and Shephard states that if K is origin symmetric then

$$1 \leqslant g(K,k;F) \leqslant \binom{n}{k}^{1/k} \leqslant \frac{cn}{k},$$

where $c > 0$ is an absolute constant. Both estimates are sharp.

Applications: II. Rogers-Shephard inequality

Theorem (G.-Markessinis-Tsolomitis)

Let K be an isotropic convex body in $\mathbb{R}^n$. For every $1 \leqslant k \leqslant n-1$ a random $F \in G_{n,k}$ satisfies

$$c_1 L_K^{-1} \sqrt{n/k} \leqslant g(K,k;F) \leqslant c_2 \sqrt{n/k} (\log n)^2 L_K$$

with probability greater than $1 - e^{-k}$, where $c_1, c_2 > 0$ are absolute constants.

Applications: II. Rogers-Shephard inequality

For the proof we show that, for any centered convex body K of volume 1 in $\mathbb{R}^n$,

- For any $1 \leqslant k \leqslant n-1$ we have

$$\int_{G_{n,k}} \frac{1}{|P_F(K)|\,|K\cap F^{\perp}|}\,d\nu_{n,k}(F) \leqslant \left(\frac{c_1\sqrt{k}}{\sqrt{n}}\right)^{k}\left(\int_{G_{n,k}} \frac{1}{|K\cap F^{\perp}|^{\frac{n}{n-k}}}\,d\nu_{n,k}(F)\right)^{\frac{n-k}{n}},$$

  where $c_1>0$ is an absolute constant.

- For any $1 \leqslant k \leqslant n-1$ we have

$$\int_{G_{n,k}} \left(|P_F(K)|\,|K\cap F^{\perp}|\right)^{1/2} d\nu_{n,k}(F) \leqslant \left(\frac{c_2 w(K)}{\sqrt{k}}\right)^{k/2},$$

  where $c_2>0$ is an absolute constant.

Applications: II. Rogers-Shephard inequality

These two inequalities are general. However, for an isotropic convex body K in $\mathbb{R}^n$ we know that:

- For any $1 \leqslant k \leqslant n-1$ and any $F \in G_{n,k}$,

$$|K \cap F^{\perp}| \geqslant \left(\frac{c_3}{L_K}\right)^k,$$

  where $c_3 > 0$ is an absolute constant.

- By E. Milman's theorem,

$$w(K) \leqslant c_4 \sqrt{n} (\log n)^2 L_K,$$

  where $c_4 > 0$ is an absolute constant.

This additional information completes the proof of the Theorem.

Applications: III. Random polytopes

- Let μ be an isotropic log-concave measure on $\mathbb{R}^n$. For every $N \geqslant n$ we consider N independent random points $x_1, \dots, x_N$ distributed according to μ and define the random polytope

$$K_N := \mathrm{conv}\{\pm x_1, \dots, \pm x_N\}.$$

- It was proved by Dafnis, G. and Tsolomitis that one can compare K_N with the L_q-centroid body of μ for a suitable value of q; roughly speaking, K_N is close to the body $Z_{\log(N/n)}(\mu)$ with high probability.

Applications: III. Random polytopes

- More precisely, given any isotropic log-concave measure μ on $\mathbb{R}^n$ and any $cn \leqslant N \leqslant e^n$, the random polytope K_N satisfies, with high probability, the inclusion

$$K_N \supseteq c_1 Z_{\log(N/n)}(\mu).$$

- On the other hand, for every $\alpha > 1$ and $q \geqslant 1$,

$$\mathbb{E}\left[\sigma(\{\theta : h_{K_N}(\theta) \geqslant \alpha h_{Z_q(\mu)}(\theta)\})\right] \leqslant N\alpha^{-q}.$$

- This estimate is sufficient for some sharp upper bounds: for all $n \leqslant N \leqslant \exp(n)$ one has

$$\mathbb{E}\left[w(K_N)\right] \leqslant c_6\, w(Z_{\log N}(\mu)).$$

Applications: III. Random polytopes

For every $1 \leqslant k \leqslant n$ consider the normalized quermassintegrals of a convex body K:

$$Q_k(K) = \left(\frac{W_{n-k}(K)}{\omega_n}\right)^{1/k} = \left(\frac{1}{\omega_k}\int_{G_{n,k}} |P_F(K)|\, d\nu_{n,k}(F)\right)^{1/k}.$$

Theorem (Dafnis-G.-Tsolomitis)

If $n^2 \leqslant N \leqslant \exp(cn)$ *then for every* $1 \leqslant k \leqslant n$ *we have*

$$L_\mu^{-1}\sqrt{\log N} \lesssim \mathbb{E}\left[Q_k(K_N)\right] \lesssim w(Z_{\log N}(K)).$$

In the range $n^2 \leqslant N \leqslant \exp(\sqrt{n})$ *one has an asymptotic formula: for every* $1 \leqslant k \leqslant n$,

$$\mathbb{E}\left[Q_k(K_N)\right] \simeq \sqrt{\log N}.$$

Applications: III. Random polytopes

Theorem (G.-Hioni-Tsolomitis)

Let $x_1, \ldots, x_N$ be independent random points distributed according to an isotropic log-concave measure μ on $\mathbb{R}^n$, and consider the random polytope $K_N := \mathrm{conv}\{\pm x_1, \ldots, \pm x_N\}$. For all $\exp(\sqrt{n}) \leqslant N \leqslant \exp(n)$ and $s \geqslant 1$ we have

$$L_\mu^{-1}\sqrt{\log N} \leqslant Q_k(K_N) \leqslant c_2(s)\sqrt{\log N}\,(\log\log N)^2,$$

for all $1 \leqslant k < n$, with probability greater than $1 - N^{-s}$.

Applications: III. Random polytopes

For any convex body C in $\mathbb{R}^n$ and any $1 \leqslant k \leqslant n$, the k-th mean outer radius of C is defined by $\tilde{R}_k(C) = \int_{G_{n,k}} R(P_F(C))\, d\nu_{n,k}(F)$.

Theorem (Alonso-Gutiérrez, Dafnis, Hernández-Cifre and Prochno)

If $n \leqslant N \leqslant \exp(\sqrt{n})$ then a random K_N satisfies, for all $1 \leqslant k \leqslant n$,

$$c \max\left\{\sqrt{k}, \sqrt{\log(N/n)}\right\} \leqslant \tilde{R}_k(K_N) \leqslant C \max\left\{\sqrt{k}, \sqrt{\log N}\right\}.$$

Theorem (G.-Hioni-Tsolomitis)

If $\exp(\sqrt{n}) \leqslant N \leqslant \exp(n)$ then a random K_N satisfies, for all $1 \leqslant k \leqslant n$,

$$c \max\left\{L_\mu^{-1}\sqrt{\log N}, \sqrt{k}, \sqrt{k/n}R(Z_{\log N}(\mu))\right\}$$
$$\leqslant \tilde{R}_k(K_N) \leqslant C \max\{\sqrt{\log N}(\log\log N)^2, \sqrt{k/n}\log N\}.$$

Applications: III. Random polytopes

Consider the average diameter of k-dimensional sections of a convex body C with $0 \in \mathrm{int}(C)$, defined by $\tilde{D}_k(C) = \int_{G_{n,k}} R(C \cap F)\, d\nu_{n,k}(F)$.

Theorem (G.-Hioni-Tsolomitis)

Given $0 < a < b < 1$, for any $an \leqslant k \leqslant bn$, a random K_N satisfies with probability greater than $1 - \exp(-c_a\sqrt{n})$:

(i) *If $n^2 \leqslant N \leqslant \exp(\sqrt{n})$ then*

$$c_a \max\left\{\frac{\sqrt{\log N}}{\log^2 n}, 1\right\} \leqslant \tilde{D}_k(K_N) \leqslant c_b\sqrt{\log N},$$

(ii) *If $\exp(\sqrt{n}) \leqslant N \leqslant \exp(n)$ then*

$$c_a \frac{\sqrt{\log N}}{\log^2 n} \leqslant \tilde{D}_k(K_N) \leqslant c_b\sqrt{\log N}(\log\log N)^2,$$

where $c_a, c_b > 0$ depend only on a and b respectively.