Mean-width and mean-norm of isotropic convex bodies

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We assume that $K$ is a centrally symmetric convex body of volume 1 in $\mathbb{R}^n$:

$$K = \{ x \in \mathbb{R}^n : \|x\| \leq 1 \}.$$

The mean-norm of $K$ is defined by

$$M(K) = \int_{S^{n-1}} \|x\| \, d\sigma(x).$$
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The support function of $K$ is

$$h_K(x) = \|x\|_* = \max\{ \langle x, y \rangle : y \in K \},$$

and the mean-width of $K$ is

$$M^*(K) = w(K) = \int_{S^{n-1}} h_K(x) \, d\sigma(x).$$
Lower bounds

- Using integration in polar coordinates and Hölder’s inequality we get

\[ M(K) \geq \left( \frac{|B_2^n|}{|K|} \right)^{1/n} \geq \frac{c_1}{\sqrt{n}}. \]

- From Urysohn’s inequality,

\[ M^*(K) \geq \text{vrad}(K) := \left( \frac{|K|}{|B_2^n|} \right)^{1/n} \geq c_2 \sqrt{n}. \]
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These lower bounds for \( M \) and \( M^* \) are sharp: if \( D_n = r_n B_2^n \) has volume 1 then \( r_n \approx \sqrt{n} \) and

\[ M(D_n) = \frac{1}{r_n} \approx \frac{1}{\sqrt{n}} \quad \text{while} \quad M^*(D_n) = r_n \approx \sqrt{n}. \]
Theorem (Lewis, Figiel-Tomczak, Pisier)

Every centrally symmetric convex body $K$ in $\mathbb{R}^n$ has a linear image (a position) $\tilde{K}$ of volume 1 such that

$$M(\tilde{K})M^*(\tilde{K}) \leq c_1 \log[d(X_K, \ell_2^n) + 1] \leq c_2 \log n.$$
Every centrally symmetric convex body $K$ in $\mathbb{R}^n$ has a linear image (a position) $\tilde{K}$ of volume 1 such that

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For this position of $K$, using the previous lower bounds, we have

$$M(\tilde{K}) \leq \frac{c \log n}{\sqrt{n}}$$

and

$$M^*(\tilde{K}) \leq c \sqrt{n} \log n.$$ 

**Question:** What can we say about the isotropic position?
Isotropic convex bodies

A convex body $K$ in $\mathbb{R}^n$ is called isotropic if it has volume 1, it is centered, and there exists a constant $L_K > 0$ such that

$$\int_K \langle x, \theta \rangle^2 \, dx = L_K^2$$

for every $\theta \in S^{n-1}$.

Hyperplane conjecture

There exists an absolute constant $C > 0$ such that $L_K \leq C$ for every $n$ and every isotropic convex body $K$ in $\mathbb{R}^n$.

Bourgain: $L_K \leq c_4 \sqrt{n \log n}$, Klartag: $L_K \leq c_4 \sqrt{n \log n}$.
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Hyperplane conjecture

There exists an absolute constant $C > 0$ such that $L_K \leq C$ for every $n$ and every isotropic convex body $K$ in $\mathbb{R}^n$.

Bourgain: $L_K \leq c \sqrt[4]{n} \log n$, Klartag: $L_K \leq c \sqrt[4]{n}$. 
Log-concave measures

A measure $\mu$ on $\mathbb{R}^n$ is called log-concave if

$$\mu(\lambda A + (1 - \lambda)B) \geq \mu(A)^{\lambda} \mu(B)^{1-\lambda}$$

for any non-empty compact subsets $A$ and $B$ of $\mathbb{R}^n$ and any $\lambda \in (0, 1)$. 

Isotropic log-concave measures

We say that a log-concave probability measure $\mu$ is isotropic if $\bar{\mu} = 0$ and $\text{Cov}(\mu)$ is the identity matrix:

$$\int x_i x_j f \mu(x) dx = \delta_{ij}.$$ 

Then, the isotropic constant of $\mu$ is

$$L_{\mu} = \|f \mu\|_{1/n} \approx f \mu(0)^{1/n}.$$
A measure $\mu$ on $\mathbb{R}^n$ is called log-concave if

$$\mu(\lambda A + (1 - \lambda)B) \geq \mu(A)\lambda \mu(B)^{1-\lambda}$$

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Then, the isotropic constant of $\mu$ is $L_\mu = \|f_{\mu}\|_\infty^{1/n} \approx f_{\mu}(0)^{1/n}$. 

(Bedlewo 2014)
Isotropic log-concave measures

If $K$ is a convex body in $\mathbb{R}^n$, then the Brunn-Minkowski inequality implies that $1_K$ is the density of a log-concave measure. $K$ is isotropic if and only if the measure $\mu_K$ with density $L^n_K \frac{1}{L_K} 1_K$ is isotropic.
If $K$ is a convex body in $\mathbb{R}^n$, then the Brunn-Minkowski inequality implies that $\mathbf{1}_K$ is the density of a log-concave measure. $K$ is isotropic if and only if the measure $\mu_K$ with density $L^n_K \frac{1}{L_K} \mathbf{1}_K$ is isotropic.

**Marginal**

The marginal of $\mu$ with respect to $F \in G_{n,k}$ is defined by

$$\pi_F \mu(A) := \mu(P_F^{-1}(A)) = \mu(A + F^\perp)$$

for all Borel subsets of $F$. The density of $\pi_F \mu$ is the function

$$f_{\pi_F \mu}(x) = \int_{x+F^\perp} f_\mu(y) \, dy, \quad x \in F.$$  

If $\mu$ is centered, log-concave or isotropic, then $\pi_F \mu$ is respectively also centered, log-concave or isotropic.
If $\mu$ is a probability measure on $\mathbb{R}^n$, the $L_q$-centroid body $Z_q(\mu)$, $q \geq 1$, is the symmetric convex body with support function

$$h_{Z_q(\mu)}(y) := \|\langle \cdot, y \rangle\|_{L^q(\mu)} = \left( \int |\langle x, y \rangle|^q d\mu(x) \right)^{1/q}.$$
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- $\mu$ is isotropic if and only if it is centered and $Z_2(\mu) = B_2^n$.
- From Hölder’s inequality it follows that $Z_2(\mu) \subseteq Z_p(\mu) \subseteq Z_q(\mu)$ for all $2 \leq p \leq q < \infty$.
- From Borell’s lemma, $Z_q(\mu) \subseteq c \frac{q}{p} Z_p(\mu)$ for all $2 \leq p < q$.
- If $\mu$ is isotropic, then $R(Z_q(\mu)) := \max\{h_{Z_q(\mu)}(\theta) : \theta \in S^{n-1}\} \leq cq$. 

(Bedlewo 2014)
If $K$ is a convex body of volume 1 in $\mathbb{R}^n$, the $L_q$-centroid body $Z_q(K)$, $q \geq 1$, is the symmetric convex body with support function

$$h_{Z_q(K)}(y) := \left( \int_K |\langle x, y \rangle|^q dx \right)^{1/q}.$$

- $K$ is isotropic if and only if it is centered and $Z_2(K) = L_K B_2^n$.

- If $K$ is centrally symmetric then

$$cK \subseteq Z_q(K) \subseteq K$$

for all $q \geq n$.

- If $K$ is isotropic and if $\mu_K$ is the isotropic measure with density $L_K^n \frac{1}{L_K} 1_{L_K}$, then

$$Z_q(K) = L_K Z_q(\mu_K).$$
The two questions

Assume that $K$ is centrally symmetric and isotropic in $\mathbb{R}^n$.

**Question 1**

To give an upper bound for $M^*(K)$.

From the inclusion $K \subseteq (n + 1)L_K B_2^n$, one has the obvious bound $M^*(K) \leq (n + 1)L_K$.

Until recently, it was known that $M^*(K) \leq cn^{3/4}L_K$.

Several approaches: Hartzoulaki, Pivovarov, “$Z_q$-bound”.

**Question 2**

To give an upper bound for $M(K)$.

From the inclusion $K \supseteq L_K B_2^n$, one has the obvious bound $M(K) \leq 1/L_K$.

Until recently, there was no lower bound depending on $n$.
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Until recently, there was no lower bound depending on $n$. 
A first idea: Dudley’s entropy estimate

Entropy numbers

The covering number $N(K, T)$ of $K$ by $T$ is the minimal number of translates of $T$ whose union covers $K$. For any $k \geq 1$ we set

$$e_k(K, T) := \inf\{s > 0 : N(K, sT) \leq 2^k\}.$$  

The $k$-th entropy number of $K$ is $e_k(K) := e_k(K, B^n_2)$. 

(Dudley's bound)

If $K$ is a centrally symmetric convex body in $\mathbb{R}^n$ then

$$\sqrt{n} M^*(K) \leq c_1 \sum_{k \geq 1} 1/\sqrt{k} e_k(K).$$
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**Entropy numbers**

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**Dudley’s bound**

If $K$ is a centrally symmetric convex body in $\mathbb{R}^n$ then

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A first idea: Dudley’s entropy estimate

Covering numbers

If $K$ is an isotropic convex body in $\mathbb{R}^n$ then

$$\log N(K, sB_2^n) \leq C_1 \frac{n^{3/2} L_K}{s}$$

for all $s > 0$. Therefore,

$$e_k(K, B_2^n) = \inf \{s > 0 : N(K, sB_2^n) \leq 2^k \} \leq C_2 \sqrt{n L_K} \frac{n}{k}.$$
A first idea: Dudley’s entropy estimate

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If $K$ is an isotropic convex body in $\mathbb{R}^n$ then

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Then, we combine this with Dudley’s bound

$$M^*(K) \leq c_1 \sum_{k \geq 1} \frac{1}{\sqrt{k}} e_k(K, B_2^n)$$

to get $M^*(K) \leq Cn^{3/4} L_K$.  

(Bedlewo 2014)  
M and $M^*$ estimates  
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The parameter $v_k(K)$

For any $k \geq 1$ we set

$$v_k(K) := \sup \{ \text{vrad}(P_F(K)) : F \in G_{n,k} \}.$$
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For any $k \geq 1$ we set

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$v_k$ and $e_k$

For every $F \in G_{n,k}$ we have

$$|P_F(K)| \leq N(P_F(K), e_k P_F(B_2^n)) |e_k B_F|$$

$$\leq N(K, e_k(2^n) B_2^n) e_k^k |B_F| \leq (2 e_k)^k |B_F|,$$

therefore

$$v_k(K) \leq 2 e_k(K).$$
A refinement by V. Milman and Pisier

Theorem (V. Milman-Pisier)

For every centrally symmetric convex body $K$ in $\mathbb{R}^n$ one has

$$\sqrt{n}M^*(K) \leq c_2 \sum_{k=1}^{n} \frac{1}{\sqrt{k}} \text{Rad}_k(K)v_k(K).$$
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Rad$_k$(K)

In the statement above, $\text{Rad}_k(K) := \sup\{\text{Rad}(X_{PF}(K)) : F \in G_{n,k}\}$ where $\text{Rad}(Y) \leq c_3 \log(d(Y, \ell_2^{\dim(Y)}) + 1)$ is the Rademacher constant of $Y$. 
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Rad$_k$(K)

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So, roughly speaking,

$$\sqrt{n} M^*(K) \lesssim \sum_{k=1}^{n} \frac{1}{\sqrt{k}} v_k(K).$$
We know that if $K$ is a centrally symmetric isotropic convex body in $\mathbb{R}^n$ then

$$Z_n(K) \simeq K.$$
Mean width: isotropic case

- We know that if $K$ is a centrally symmetric isotropic convex body in $\mathbb{R}^n$ then
  
  $$Z_n(K) \cong K.$$ 

- So, we look for an upper bound for $M^*(Z_n(K))$, and more generally for $M^*(Z_q(K))$. To this end we use
  
  $$\sqrt{n}M^*(Z_q(K)) \lesssim \sum_{k=1}^{n} \frac{1}{\sqrt{k}} v_k(Z_q(K)).$$
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$$\sqrt{n}M^*(Z_q(K)) \lesssim \sum_{k=1}^{n} \frac{1}{\sqrt{k}} v_k(Z_q(K)).$$

In order to estimate $v_k(Z_q(K))$ we consider any $F \in G_{n,k}$ and try to give an upper bound for

$$v_{rad}(P_F(Z_q(K))) = \left( \frac{|P_F(Z_q(K))|}{|B^k_2|} \right)^{1/k}.$$
Mean width: bound for $v_k(Z_q(K))$

### Projections of $P_F(\mu)$

For any $F \in G_{n,k}$ we have $P_F(Z_q(\mu)) = Z_q(\pi_F(\mu))$.

Since $\pi_F(\mu)$ is an isotropic log-concave measure on $F$ we may use:

Theorem (Paouris)

If $\nu$ is an isotropic log-concave measure on $\mathbb{R}^k$ then

$$v_{\text{rad}}(Z_q(\nu)) \leq c_5 \sqrt{q}$$

if $q \leq k$ and

$$v_{\text{rad}}(Z_q(\nu)) \leq c_5 \left(\frac{q}{k}\right) \sqrt{k}$$

if $q \geq k$.

Applied to $\mu = \mu_K$, this gives

$$v_k(Z_q(K)) \leq c_6 \sqrt{q} \max(\sqrt{q}, \sqrt{k}) L_K.$$
Mean width: bound for $v_k(Z_q(K))$

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$$v_{\text{rad}}(Z_q(\nu)) \leq c_5 (q/k) \sqrt{k} \quad \text{if } q \geq k$$
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- Applied to $\mu = \mu_K$, this gives

$$v_k(Z_q(K)) \leq c_6 \sqrt{\frac{q}{k}} \max(\sqrt{q}, \sqrt{k}) L_K.$$
Theorem (E. Milman)

For every centrally symmetric isotropic convex body $K$ in $\mathbb{R}^n$ and for every $2 \leq q \leq n$,

$$M^*(Z_q(K)) \leq c\sqrt{q}(\log q)^2 L_K.$$ 

In particular,

$$M^*(K) \leq c\sqrt{n}(\log n)^2 L_K.$$
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In particular,

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For the proof we write

$$L_K^{-1} \sqrt{n} M^*(Z_q(K)) \sim L_K^{-1} \sum_{k=1}^{n} \frac{1}{\sqrt{k}} v_k(z_q(K)) \leq c_2 \sum_{k=1}^{n} \max \left( \sqrt{\frac{q}{k}}, \frac{q}{k} \right)$$

$$\simeq q \sum_{k=1}^{q} \frac{1}{k} + \sqrt{q} \sum_{k=q}^{n} \frac{1}{\sqrt{q}} \simeq q \log q + \sqrt{q}\sqrt{n} \leq \sqrt{n} \sqrt{q} \log q.$$
Let $K$ be a centrally symmetric isotropic convex body $K$ in $\mathbb{R}^n$. In order to estimate $M(K)$ we may start from the estimate of V. Milman and Pisier, using duality: we have

$$\sqrt{n}M(K) \leq c_2 \sum_{k=1}^{\infty} \frac{1}{\sqrt{k}} \text{Rad}_k(K^\circ) v_k(K^\circ).$$
Mean-norm: the dual problem

Let $K$ be a centrally symmetric isotropic convex body $K$ in $\mathbb{R}^n$. In order to estimate $M(K)$ we may start from the estimate of V. Milman and Pisier, using duality: we have

$$\sqrt{n}M(K) \leq c_2 \sum_{k=1}^{n} \frac{1}{\sqrt{k}} \text{Rad}_k(K^\circ) \nu_k(K^\circ).$$

Note that

$$\nu_k(K^\circ) := \sup \{ \text{vrad}(P_F(K^\circ)) : F \in G_{n,k} \} \approx \frac{1}{\inf \{ \text{vrad}(K \cap F) : F \in G_{n,k} \}} =: \frac{1}{w_{-k}(K)}.$$
Mean-norm: the dual problem

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$$\sqrt{n} M(K) \leq c_2 \sum_{k=1}^{n} \frac{1}{\sqrt{k}} \text{Rad}_k(K^\circ) v_k(K^\circ).$$

- Note that

$$v_k(K^\circ) := \sup \{ v_{\text{rad}}(P_F(K^\circ)) : F \in G_{n,k} \}$$

$$\simeq \frac{1}{\inf \{ v_{\text{rad}}(K \cap F) : F \in G_{n,k} \}} =: \frac{1}{w_k^-(K)}.$$

- Because of the formula $P_F(Z_q(\mu)) = Z_q(\pi_F(\mu))$ we would prefer to work with the quantity

$$v_k^-(K) := \inf \{ v_{\text{rad}}(P_F(K)) : F \in G_{n,k} \}.$$

Note that $w_k^-(K) \leq v_k^-(K)$. 

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Theorem (G.-E. Milman)

For every centrally symmetric convex body $K$ in $\mathbb{R}^n$ and $k \geq 1$ we have

$$e_k(K^\circ, B_2^n) \leq C \frac{n}{k} \log \left( e + \frac{n}{k} \right) \sup_{1 \leq m \leq \min(k, n)} \frac{1}{k^{\frac{3m}{k} \nu_m(K)}}.$$
Mean norm: new general bound

Theorem (G.-E. Milman)

For every centrally symmetric convex body $K$ in $\mathbb{R}^n$ and $k \geq 1$ we have

$$e_k(K^\circ, B_2^n) \leq C \frac{n}{k} \log \left( e + \frac{n}{k} \right) \sup_{1 \leq m \leq \min(k, n)} \frac{1}{2^{3m} v_m^{-}(K)}.$$

This leads to the next general bound:

Theorem (G.-E. Milman)

Let $K$ be a centrally symmetric convex body in $\mathbb{R}^n$ with $K \supseteq rB_2^n$. Then,

$$\sqrt{n}M(K) \leq C \sum_{k=1}^{n} \frac{1}{\sqrt{k}} \min \left( \frac{1}{r}, \frac{n}{k} \log \left( e + \frac{n}{k} \right) \frac{1}{v_k^{-}(K)} \right).$$
We need a lower bound for $v_{\text{rad}}(P_F(Z_q(\mu))) = v_{\text{rad}}(Z_q(\pi_F(\mu)))$, when $F \in G_{n,k}$. 
We need a lower bound for $v_{\text{rad}}(P_F(Z_q(\mu))) = v_{\text{rad}}(Z_q(\pi_F(\mu)))$, when $F \in G_{n,k}$.

The main tool is a theorem of Klartag and E. Milman: if $\nu$ is an isotropic log-concave measure on $\mathbb{R}^k$ then, for all $1 \leq q \leq \sqrt{k}$,

$$v_{\text{rad}}(Z_q(\nu)) \geq c \sqrt{q}.$$
Mean norm of $Z_q(\mu)$

- We need a lower bound for $\text{vrad}(P_F(Z_q(\mu))) = \text{vrad}(Z_q(\pi_F(\mu)))$, when $F \in G_{n,k}$.

- The main tool is a theorem of Klartag and E. Milman: if $\nu$ is an isotropic log-concave measure on $\mathbb{R}^k$ then, for all $1 \leq q \leq \sqrt{k}$,

$$\text{vrad}(Z_q(\nu)) \geq c\sqrt{q}.$$  

- It follows that if $\mu$ is an isotropic log-concave measure on $\mathbb{R}^n$ then, for any $q \geq 1$ and $k = 1, \ldots, n$ we have

$$\nu_k^-(Z_q(\mu)) \geq c\sqrt{\min(q, \sqrt{k})}.$$  

(Bedlewo 2014)
Mean norm of $Z_q(\mu)$

Theorem (G.-E. Milman)

For any isotropic log-concave measure $\mu$ on $\mathbb{R}^n$ and any $q \leq q_0 := (n \log(e + n))^{2/5}$,

$$M(Z_q(\mu)) \leq C \frac{\sqrt{\log q}}{\sqrt[4]{q}}.$$
Mean norm of $Z_q(\mu)$

**Theorem (G.-E. Milman)**

For any isotropic log-concave measure $\mu$ on $\mathbb{R}^n$ and any $q \leq q_0 := (n \log(e + n))^{2/5}$,

$$M(Z_q(\mu)) \leq C \frac{\sqrt{\log q}}{\sqrt[4]{q}}.$$ 

Since $M(K) \leq M(Z_{q_0}(K))$ we have:

**Theorem (G.-E. Milman)**

For any centrally symmetric isotropic convex body $K$ in $\mathbb{R}^n$ we have

$$M(K) \leq \frac{C}{L_K} \frac{\log^{2/5}(e + n)}{n^{1/10}}.$$
Paouris has proved that if \( \mu \) is an isotropic log-concave measure on \( \mathbb{R}^n \), then

\[
L_\mu \sim \frac{C \sqrt{n}}{v_{\text{rad}}(Z_n(\mu))} \leq \frac{C \sqrt{n}}{v_{\text{rad}}(Z_q(\mu))}
\]

for all \( q \leq n \).

Klartag and E. Milman have proved that \( v_{\text{rad}}(Z_{\sqrt{n}}(\mu)) \geq c \sqrt[4]{n} \).

This gives a proof of the best known bound \( L_\mu \leq C \sqrt[4]{n} \).

**Question**

To prove that \( M^*(Z_q(\mu)) \geq c \sqrt{q} \) for \( q \gg \sqrt{n} \).

By Urysohn’s inequality this is “less” than showing that \( v_{\text{rad}}(Z_q(\mu)) \geq c \sqrt{q} \) for \( q \gg \sqrt{n} \).
Applications: 1. Sub-Gaussian directions

- A direction $\theta \in S^{n-1}$ is a $\psi_\alpha$-direction (where $1 \leq \alpha \leq 2$) for $K$ with constant $b > 0$ if
  \[ \| \langle \cdot, \theta \rangle \|_{\psi_\alpha} \leq b \| \langle \cdot, \theta \rangle \|_2, \]
  where
  \[ \| \langle \cdot, \theta \rangle \|_{\psi_\alpha} := \inf \left\{ t > 0 : \int_K \exp \left( \frac{|\langle x, \theta \rangle|}{t} \right) \alpha \, dx \leq 2 \right\}. \]

- It is known that
  \[ \| \langle \cdot, \theta \rangle \|_{\psi_\alpha} \simeq \sup_{q \geq \alpha} \frac{\| \langle \cdot, \theta \rangle \|_q}{q^{1/\alpha}}. \]
Applications: I. Sub-Gaussian directions

- From the Brunn-Minkowski inequality it follows that every $\theta \in S^{n-1}$ is a $\psi_1$-direction for $K$ with an absolute constant $C$.

- Question: is it true that there exists an absolute constant $C > 0$ such that every $K$ has at least one sub-Gaussian direction ($\psi_2$-direction) with constant $C$?

- It is known that the answer is affirmative, with a constant $O(\sqrt{\log n})$. This is due to Paouris, Valettas and G. (2011). The first result of this type was proved by Klartag (2006).

- In the isotropic case: is it true that most $\theta \in S^{n-1}$ are sub-Gaussian directions for $K$ with a constant at most logarithmic in $n$?
Theorem (Brazitikos-Hioni)

Let $K$ be an isotropic symmetric convex body in $\mathbb{R}^n$. Then,

$$\int_{S^{n-1}} \|\langle \cdot, \theta \rangle \|_{\psi_2} d\sigma(\theta) \leq C (\log n)^3 L_K,$$

where $C > 0$ is an absolute constant.
Applications: I. Sub-Gaussian directions

**Theorem (Brazitikos-Hioni)**

Let $K$ be an isotropic symmetric convex body in $\mathbb{R}^n$. Then,

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where $C > 0$ is an absolute constant.

**Theorem (Brazitikos-Hioni)**

Let $K$ be an isotropic symmetric convex body in $\mathbb{R}^n$. Then, for any $\alpha > 1$ we have

$$\|\langle \cdot, \theta \rangle\|_{\psi_2} \leq C \sqrt{\alpha} (\log n)^{3/2} \max \left\{ \frac{\sqrt{\log n} L_K}{\sqrt{\alpha}}, L_K^2 \right\}$$

for all $\theta$ in a subset $\Theta$ of $S^{n-1}$ with $\sigma(\Theta) \geq 1 - n^{-\alpha}$, where $C > 0$ is an absolute constant.
Applications: 1. Sub-Gaussian directions

- Proof of the first theorem: for any $\theta \in S^{n-1}$ we have

$$\|\langle \cdot, \theta \rangle\|_{\psi_2} \leq C_1 \max_{1 \leq s \leq m} \frac{h_{Z_{2^s}}(K)(y)}{2^{s/2}}$$

where $m = \lceil \log_2 n \rceil$. It trivially follows that

$$\mathbb{E}_{\theta}(\|\langle \cdot, \theta \rangle\|_{\psi_2}) \leq C_1 \sum_{s=1}^{m} \frac{M^*(Z_{2^s}(K))}{2^{s/2}}.$$

- We know that

$$M^*(Z_{2^s}(K)) \leq C_2 s 2^{s/2} \max \left\{ \frac{s 2^{s/2}}{\sqrt{n}}, 1 \right\} L_K.$$

- Therefore,

$$\mathbb{E}_{\theta}(\|\langle \cdot, \theta \rangle\|_{\psi_2}) \leq C_3 \sum_{s=1}^{m} s \max \left\{ \frac{s 2^{s/2}}{\sqrt{n}}, 1 \right\} L_K \leq C_3 m^3 L_K \leq C_4 (\log n)^3 L_K.$$
Applications: II. Rogers-Shephard inequality

Let $K$ be a convex body of volume 1 in $\mathbb{R}^n$ with $0 \in \text{int}(K)$. For every $1 \leq k \leq n - 1$ and any $F \in G_{n,k}$ we define

$$g(K, k; F) := \left( |P_F(K)| |K \cap F^\perp| \right)^{1/k},$$

where $F^\perp$ denotes the orthogonal subspace of $F$ in $\mathbb{R}^n$.

A classical inequality of Rogers and Shephard states that if $K$ is origin symmetric then

$$1 \leq g(K, k; F) \leq \left( \frac{n}{k} \right)^{1/k} \leq \frac{cn}{k},$$

where $c > 0$ is an absolute constant. Both estimates are sharp.
Theorem (G.-Markessinis-Tsolomitis)

Let $K$ be an isotropic convex body in $\mathbb{R}^n$. For every $1 \leq k \leq n - 1$ a random $F \in G_{n,k}$ satisfies

$$c_1 L_K^{-1} \sqrt{n/k} \leq g(K, k; F) \leq c_2 \sqrt{n/k} (\log n)^2 L_K$$

with probability greater than $1 - e^{-k}$, where $c_1, c_2 > 0$ are absolute constants.
For the proof we show that, for any centered convex body \( K \) of volume 1 in \( \mathbb{R}^n \),

- For any \( 1 \leq k \leq n - 1 \) we have
  \[
  \int_{G_{n,k}} \frac{1}{|P_F(K)||K \cap F^\perp|} d\nu_{n,k}(F)
  \leq \left( \frac{c_1 \sqrt{k}}{\sqrt{n}} \right)^k \left( \int_{G_{n,k}} \frac{1}{|K \cap F^\perp|^\frac{n}{n-k}} d\nu_{n,k}(F) \right) \frac{n-k}{n},
  \]
  where \( c_1 > 0 \) is an absolute constant.

- For any \( 1 \leq k \leq n - 1 \) we have
  \[
  \int_{G_{n,k}} (|P_F(K)||K \cap F^\perp|)^{1/2} d\nu_{n,k}(F)
  \leq \left( \frac{c_2 w(K)}{\sqrt{k}} \right)^{k/2},
  \]
  where \( c_2 > 0 \) is an absolute constant.
These two inequalities are general. However, for an isotropic convex body $K$ in $\mathbb{R}^n$ we know that:

- For any $1 \leq k \leq n - 1$ and any $F \in G_{n,k}$,
  \[
  |K \cap F^\perp| \geq \left( \frac{c_3}{L_K} \right)^k,
  \]
  where $c_3 > 0$ is an absolute constant.

- By E. Milman’s theorem,
  \[
  w(K) \leq c_4 \sqrt{n} (\log n)^2 L_K,
  \]
  where $c_4 > 0$ is an absolute constant.

This additional information completes the proof of the Theorem.
Let $\mu$ be an isotropic log-concave measure on $\mathbb{R}^n$. For every $N \geq n$ we consider $N$ independent random points $x_1, \ldots, x_N$ distributed according to $\mu$ and define the random polytope

$$K_N := \text{conv}\{\pm x_1, \ldots, \pm x_N\}.$$

It was proved by Dafnis, G. and Tsolomitis that one can compare $K_N$ with the $L_q$-centroid body of $\mu$ for a suitable value of $q$; roughly speaking, $K_N$ is close to the body $Z_{\log\left(\frac{N}{n}\right)}(\mu)$ with high probability.
More precisely, given any isotropic log-concave measure $\mu$ on $\mathbb{R}^n$ and any $cn \leq N \leq e^n$, the random polytope $K_N$ satisfies, with high probability, the inclusion

$$K_N \supseteq c_1 Z_{\log(N/n)}(\mu).$$

On the other hand, for every $\alpha > 1$ and $q \geq 1$,

$$\mathbb{E} \left[ \sigma(\{\theta : h_{K_N}(\theta) \geq \alpha h_{Z_q(\mu)}(\theta)\}) \right] \leq N\alpha^{-q}.$$

This estimate is sufficient for some sharp upper bounds: for all $n \leq N \leq \exp(n)$ one has

$$\mathbb{E} \left[ w(K_N) \right] \leq c_6 \cdot w(Z_{\log N}(\mu)).$$
For every $1 \leq k \leq n$ consider the normalized quermassintegrals of a convex body $K$:

$$Q_k(K) = \left( \frac{W_{n-k}(K)}{\omega_n} \right)^{1/k} = \left( \frac{1}{\omega_k} \int_{G_{n,k}} |P_F(K)| \, d\nu_{n,k}(F) \right)^{1/k}.$$ 

**Theorem (Dafnis-G.-Tsolomitis)**

If $n^2 \leq N \leq \exp(cn)$ then for every $1 \leq k \leq n$ we have

$$L_\mu^{-1} \sqrt{\log N} \lesssim \mathbb{E} \left[ Q_k(K_N) \right] \lesssim w(Z_{\log N}(K)).$$

In the range $n^2 \leq N \leq \exp(\sqrt{n})$ one has an asymptotic formula: for every $1 \leq k \leq n$,

$$\mathbb{E} \left[ Q_k(K_N) \right] \simeq \sqrt{\log N}.$$
Applications: III. Random polytopes

Theorem (G.-Hioni-Tsolomitis)

Let $x_1, \ldots, x_N$ be independent random points distributed according to an isotropic log-concave measure $\mu$ on $\mathbb{R}^n$, and consider the random polytope $K_N := \text{conv}\{\pm x_1, \ldots, \pm x_N\}$. For all $\exp(\sqrt{n}) \leq N \leq \exp(n)$ and $s \geq 1$ we have

$$L_{\mu}^{-1} \sqrt{\log N} \leq Q_k(K_N) \leq c_2(s) \sqrt{\log N} \left(\log \log N\right)^2,$$

for all $1 \leq k < n$, with probability greater than $1 - N^{-s}$. (Bedlewo 2014)
For any convex body $C$ in $\mathbb{R}^n$ and any $1 \leq k \leq n$, the $k$-th mean outer radius of $C$ is defined by $\tilde{R}_k(C) = \int_{G_{n,k}} R(P_F(C)) \, d\nu_{n,k}(F)$.

**Theorem (Alonso-Gutiérrez, Dafnis, Hernández-Cifre and Prochno)**

If $n \leq N \leq \exp(\sqrt{n})$ then a random $K_N$ satisfies, for all $1 \leq k \leq n$,

$$c \max \left\{ \sqrt{k}, \sqrt{\log(N/n)} \right\} \leq \tilde{R}_k(K_N) \leq C \max \left\{ \sqrt{k}, \sqrt{\log N} \right\}.$$

**Theorem (G.-Hioni-Tsolomitis)**

If $\exp(\sqrt{n}) \leq N \leq \exp(n)$ then a random $K_N$ satisfies, for all $1 \leq k \leq n$,

$$c \max \left\{ L_{\mu}^{-1} \sqrt{\log N}, \sqrt{k}, \sqrt{k/nR(Z_{\log N}(\mu))} \right\} \leq \tilde{R}_k(K_N) \leq C \max \left\{ \sqrt{\log N (\log \log N)^2}, \sqrt{k/n \log N} \right\}.$$
Consider the average diameter of $k$-dimensional sections of a convex body $C$ with $0 \in \text{int}(C)$, defined by $\tilde{D}_k(C) = \int_{G_{n,k}} R(C \cap F) \, d\nu_{n,k}(F)$.

**Theorem (G.-Hioni-Tsolomitis)**

Given $0 < a < b < 1$, for any $an \leq k \leq bn$, a random $K_N$ satisfies with probability greater than $1 - \exp(-c_a \sqrt{n})$:

(i) If $n^2 \leq N \leq \exp(\sqrt{n})$ then

$$c_a \max \left\{ \frac{\sqrt{\log N}}{\log^2 n}, 1 \right\} \leq \tilde{D}_k(K_N) \leq c_b \sqrt{\log N},$$

(ii) If $\exp(\sqrt{n}) \leq N \leq \exp(n)$ then

$$c_a \frac{\sqrt{\log N}}{\log^2 n} \leq \tilde{D}_k(K_N) \leq c_b \sqrt{\log N} (\log \log N)^2,$$

where $c_a, c_b > 0$ depend only on $a$ and $b$ respectively.