Indecomposable extensions of separable Banach spaces

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An on-going project

This talk is a report on an on-going project that has involved S. Argyros, D. Freeman, the late E. Odell, Th. Raikoftsalis, Th. Schlumprecht and D. Zisimopolou, as well as the speaker.

We are interested in the question: *which separable Banach spaces Y admit an embedding into a separable, indecomposable Banach space X?* .

Recall that a Banach space X is *decomposable* if there exists a bounded linear projection $P : X \to X$ with infinite-dimensional image and kernel, and *indecomposable* otherwise.

A natural (and optimistic) conjecture is that such an embedding exist whenever Y has no subspace isomorphic to c_0.

Obviously, if true, this would be the best possible result since, by Sobczyk's theorem, any separable space X containing c_0 is decomposable.

Few operators and very few operators

As well as indecomposability, we are interested in the stronger properties of having few, or even very few, operators.

We say that X has *few* operators if every $T \in \mathscr{L}(X)$ has the form $\lambda I + S$, with λ a scalar and S strictly singular, and *very few* operators if, moreover, all strictly singular operators on X are compact. A space with very few operators is also said to have the *scalar-plus-compact* property.

Spaces with few operators have been known since the work of Gowers and Maurey (1993). In fact, most constructions of *hereditarily indecomposable*, or HI, spaces result in this stronger property.

The first space with the scalar-plus-compact property appeared in 2011 (Argyros–Haydon).

The role of BD-constructions

The method used in the scalar-plus-compact paper (AH) involved (a generalization of) a construction of exotic $\mathscr{L}_\infty$-spaces due to Bourgain and Delbaen (1980). I shall refer to such things as "BD-constructions", and say a bit more about them later.

At about the same time, another BD construction was used by Freeman, Odell and Schlumprecht to prove an unexpected embedding theorem:

(FOS 2011) *If Y^* is separable then Y embeds in to a space X with X^* isomorphic to ℓ_1.*

The FOS paper also introduced the technique of "augmentation" for BD-spaces, something that has played a crucial role in subsequent developments.

Indecomposable extensions when Y^* is separable

The first general result about indecomposable extensions, combining ideas from (AH) and (FOS), is due to the seven authors mentioned at the beginning of the talk and was published in 2012.

(AFHORSZ) *If Y is separable and super-reflexive then Y embeds in a separable space X with the scalar-plus-compact property.*

As yet unpublished, and part of the "on-going project", is the following, which represents the limit of what we can do at present in the case of a space with separable dual.

Theorem (AFHORSZ)

If Y^* is separable and c_0 does not embed into Y^{**} then Y embeds in a separable space X with the scalar-plus-compact property.

The c_0 condition.

The assumption we have had to make in the last theorem, namely that c_0 does not embed in Y^{**} is (unfortunately) a much stronger condition than non-embeddability of c_0 in Y itself.

The way we use this strong hypothesis is via the following lemma, which uses some old results of Pełczyński.

Lemma

Let X be a separable Banach space and let Y be a subspace of X such that Y^{**} does not contain c_0. Assume that Z^* is isomorphic to ℓ_1 and that $V : Z \to X$ is an operator. If $QV : Z \to X/Y$ is compact then so is V.

We call this the "Quotient-Compact Property."

A result for spaces with non-separable dual

Until recently, we had one result about spaces with non-separable dual. It is a very special case:

The space ℓ_1 embeds into a space with very few operators.

But I want to talk today about a theorem that establishes a weaker result about a more general class of spaces.

Theorem (AFHORSZ)

If Y is separable and c_0 does not embed in Y^{**} then Y embeds into a space with few operators.

A nice embedding for separable Banach spaces

I should like to give an idea of how we prove this theorem, avoiding technical aspects. In particular, I shall not go into details about what a BD-construction is nor into how the augmentation procedure works.

We start by introducing the compact space $2^{\leq\omega}$ consisting of all finite or infinite sequences of 0's and 1's. It is the union of the Cantor set 2^{ω} with the discrete dyadic tree $2^{<\omega}$. Note that the dual space $\mathscr{C}(2^{\leq\omega})^*$ is the direct sum $\ell_1(2^{<\omega}) \oplus \mathscr{M}(2^{\omega})$.

Lemma

Any separable Banach space Y admits an isomorphic embedding into $\mathscr{C}(2^{\leq\omega})$ in such a way that $\ell_1(2^{<\omega}) \cap Y^{\perp}$ is a norming subspace for $\mathscr{C}(2^{\leq\omega})/Y$.

Embedding in a BD-space

By starting with Y as a subspace of $\mathscr{C}(2^{\leq\omega})$ and applying an "augmentation", we obtain the following.

Theorem

Let Y be an arbitrary separable Banach space. There exist a separable $\mathscr{L}_\infty$-space X containing Y, and a quotient operator $R : X \to C = \mathscr{C}(2^\omega)$ with the following properties:

1. $(\ker R)^*$ *is isomorphic to* ℓ_1;
2. X/Y *is an asymptotic* ℓ_1*-space (in particular, does not contain* c_0*);*
3. *every operator* $V : \ker R \to X/Y$ *has the form* $\lambda Q\restriction_{\ker R} + K$, *where* λ *is a scalar,* K *is compact and* Q *is the quotient map from* X *to* X/Y.

Proving that X has few operators

We now want to show that the space X of the above theorem has few operators, provided that Y^{**} does not contain c_0.

Consider any $T \in \mathscr{L}(X)$. The operator $QT \restriction_{\ker R}: \ker R \to X/Y$ may be written $\lambda Q \restriction_{\ker R} + K_1$, where K_1 is compact.

So $V = (T - \lambda I) \restriction_{\ker} R : \ker R \to X$ has the property that QV is compact. So by the Compact Quotient Property, V itself is compact.

By a theorem of Lindenstrauss, the compact operator V from $\ker R.T$ to the $\mathscr{L}_\infty$-space X has a compact extension $K : X \to X$.

We see now that $T - \lambda I - K$ is an operator on X that is 0 on $\ker R$. So we can write $T = \lambda I + K + UR$, where $U : C \to X$.

To wrap things up we note that since Y and X/Y do not contain c_0, neither does X, and so $U : C \to X$ is automatically weakly compact, and strictly singular.

Can we improve “few” to “very few”?

Certainly not without an additional hypothesis.For instance, if the separable space X contains $\ell_1 \oplus \ell_2$ then X cannot have very few operators. Indeed, by a theorem of Pełczyński, there is a quotient operator R from X onto C, and of course there is a quotient operator $Q : C \to \ell_2$.The composition QR considered as an operator from X to itself is strictly singular and not compact.

More generally, if Y contains ℓ_1 and there is a non-compact operator from C into Y then Y cannot embed into a separable space with very few opertors. We do, however, have a positive result.

Theorem

If Y is separable, Y^{**} does not contain c_0 and every operator from C to Y is compact then Y embeds in a separable space X with very few operators.

A three-space result

In fact this follows from what we have just done thanks to the following three-space result.

Proposition

Let $C = \mathscr{C}(2^\omega)$ (or $\mathscr{C}[0,1]$). If every operator from C to Y and from C to X/Y is compact, then the same is true for every operator from C to X.

In the previous proof, we used the fact that neither Y nor X/Y contains c_0 to show that the operator $U : C \to X$ is *weakly* compact. With the new hypothesis, every operator from C to Y is compact, and the same is true for every operator from C to X/Y, because X/Y is asymptotic ℓ_1. So now we conclude that $U : C \to X$ is compact, and not just weakly compact, by the 3-space result.