Indecomposable extensions of separable Banach spaces

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This talk is a report on an on-going project that has involved S. Argyros, D. Freeman, the late E. Odell, Th. Raikoftsalis, Th. Schlumprecht and D. Zisimopolou, as well as the speaker.

We are interested in the question: \textit{which separable Banach spaces} $Y$ \textit{admit an embedding into a separable, indecomposable Banach space} $X$?.

Recall that a Banach space $X$ is \textit{decomposable} if there exists a bounded linear projection $P : X \to X$ with infinite-dimensional image and kernel, and \textit{indecomposable} otherwise.

A natural (and optimistic) conjecture is that such an embedding exist whenever $Y$ has no subspace isomorphic to $c_0$.

Obviously, if true, this would be the best possible result since, by Sobczyk’s theorem, any separable space $X$ containing $c_0$ is decomposable.
As well as indecomposability, we are interested in the stronger properties of having few, or even very few, operators.

We say that $X$ has few operators if every $T \in \mathcal{L}(X)$ has the form $\lambda I + S$, with $\lambda$ a scalar and $S$ strictly singular, and very few operators if, moreover, all strictly singular operators on $X$ are compact. A space with very few operators is also said to have the scalar-plus-compact property.

Spaces with few operators have been known since the work of Gowers and Maurey (1993). In fact, most constructions of hereditarily indecomposable, or HI, spaces result in this stronger property.

The first space with the scalar-plus-compact property appeared in 2011 (Argyros–Haydon).
The role of BD-constructions

The method used in the scalar-plus-compact paper (AH) involved (a generalization of) a construction of exotic $L_\infty$-spaces due to Bourgain and Delbaen (1980). I shall refer to such things as "BD-constructions", and say a bit more about them later.

At about the same time, another BD construction was used by Freeman, Odell and Schlumprecht to prove an unexpected embedding theorem:

(FOS 2011) *If $Y^*$ is separable then $Y$ embeds in to a space $X$ with $X^*$ isomorphic to $\ell_1$.*

The FOS paper also introduced the technique of “augmentation” for BD-spaces, something that has played a crucial role in subsequent developments.
The first general result about indecomposable extensions, combining ideas from (AH) and (FOS), is due to the seven authors mentioned at the beginning of the talk and was published in 2012.

(AFHORSZ) If $Y$ is separable and super-reflexive then $Y$ embeds in a separable space $X$ with the scalar-plus-compact property.

As yet unpublished, and part of the “on-going project”, is the following, which represents the limit of what we can do at present in the case of a space with separable dual.

**Theorem (AFHORSZ)**

If $Y^*$ is separable and $c_0$ does not embed into $Y^{**}$ then $Y$ embeds in a separable space $X$ with the scalar-plus-compact property.
The assumption we have had to make in the last theorem, namely that $c_0$ does not embed in $Y^{**}$ is (unfortunately) a much stronger condition than non-embeddability of $c_0$ in $Y$ itself.

The way we use this strong hypothesis is via the following lemma, which uses some old results of Pełczyński.

**Lemma**

Let $X$ be a separable Banach space and let $Y$ be a subspace of $X$ such that $Y^{**}$ does not contain $c_0$. Assume that $Z^*$ is isomorphic to $\ell_1$ and that $V : Z \to X$ is an operator. If $QV : Z \to X/Y$ is compact then so is $V$.

We call this the “Quotient-Compact Property.”
Until recently, we had one result about spaces with non-separable dual. It is a very special case:

*The space $\ell_1$ embeds into a space with very few operators.*

But I want to talk today about a theorem that establishes a weaker result about a more general class of spaces.

**Theorem (AFHORSZ)**

If $Y$ is separable and $c_0$ does not embed in $Y^{**}$ then $Y$ embeds into a space with few operators.
I should like to give an idea of how we prove this theorem, avoiding technical aspects. In particular, I shall not go into details about what a BD-construction is nor into how the augmentation procedure works.

We start by introducing the compact space $2^{\leq \omega}$ consisting of all finite or infinite sequences of 0’s and 1’s. It is the union of the Cantor set $2^{\omega}$ with the discrete dyadic tree $2^{<\omega}$. Note that the dual space $C(2^{\leq \omega})^*$ is the direct sum $\ell_1(2^{<\omega}) \oplus M(2^{\omega})$.

Lemma

Any separable Banach space $Y$ admits an isomorphic embedding into $C(2^{\leq \omega})$ in such a way that $\ell_1(2^{<\omega}) \cap Y^\perp$ is a norming subspace for $C(2^{\leq \omega})/Y$. 
Embedding in a BD-space

By starting with $Y$ as a subspace of $\mathcal{C}(2^{\leq \omega})$ and applying an “augmentation”, we obtain the following.

**Theorem**

Let $Y$ be an arbitrary separable Banach space. There exist a separable $L_\infty$-space $X$ containing $Y$, and a quotient operator $R : X \to C = \mathcal{C}(2^\omega)$ with the following properties:

1. $(\ker R)^* \text{ is isomorphic to } \ell_1$;
2. $X / Y$ is an asymptotic $\ell_1$-space (in particular, does not contain $c_0$);
3. every operator $V : \ker R \to X / Y$ has the form $\lambda Q \upharpoonright_{\ker R} + K$, where $\lambda$ is a scalar, $K$ is compact and $Q$ is the quotient map from $X$ to $X / Y$. 
Proving that $X$ has few operators

We now want to show that the space $X$ of the above theorem has few operators, provided that $Y^{**}$ does not contain $c_0$.

Consider any $T \in \mathcal{L}(X)$. The operator $QT \upharpoonright_{\ker R} : \ker R \to X/Y$ may be written $\lambda Q \upharpoonright_{\ker R} + K_1$, where $K_1$ is compact.

So $V = (T - \lambda I) \upharpoonright_{\ker R} : \ker R \to X$ has the property that $QV$ is compact. So by the Compact Quotient Property, $V$ itself is compact.

By a theorem of Lindenstrauss, the compact operator $V$ from $\ker R.T$ to the $\mathcal{L}_\infty$-space $X$ has a compact extension $K : X \to X$.

We see now that $T - \lambda I - K$ is an operator on $X$ that is 0 on $\ker R$. So we can write $T = \lambda I + K + UR$, where $U : C \to X$.

To wrap things up we note that since $Y$ and $X/Y$ do not contain $c_0$, neither does $X$, and so $U : C \to X$ is automatically weakly compact, and strictly singular.
Can we improve “few” to “very few”?

Certainly not without an additional hypothesis. For instance, if the separable space $X$ contains $\ell_1 \oplus \ell_2$ then $X$ cannot have very few operators. Indeed, by a theorem of Pełczyński, there is a quotient operator $R$ from $X$ onto $C$, and of course there is a quotient operator $Q : C \rightarrow \ell_2$. The composition $QR$ considered as an operator from $X$ to itself is strictly singular and not compact.

More generally, if $Y$ contains $\ell_1$ and there is a non-compact operator from $C$ into $Y$ then $Y$ cannot embed into a separable space with very few operators. We do, however, have a positive result.

**Theorem**

*If $Y$ is separable, $Y^{**}$ does not contain $c_0$ and every operator from $C$ to $Y$ is compact then $Y$ embeds in a separable space $X$ with very few operators.*
In fact this follows from what we have just done thanks to the following three-space result.

**Proposition**

Let $C = C(2^\omega)$ (or $C[0,1]$). If every operator from $C$ to $Y$ and from $C$ to $X/Y$ is compact, then the same is true for every operator from $C$ to $X$.

In the previous proof, we used the fact that neither $Y$ nor $X/Y$ contains $c_0$ to show that the operator $U : C \to X$ is *weakly* compact. With the new hypothesis, every operator from $C$ to $Y$ is compact, and the same is true for every operator from $C$ to $X/Y$, because $X/Y$ is asymptotic $\ell_1$. So now we conclude that $U : C \to X$ is compact, and not just weakly compact, by the 3-space result.