Hyperplane inequalities for measures of convex bodies.

Alexander Koldobsky

University of Missouri-Columbia

Hyperplane problem

Hyperplane problem: Does there exist an absolute constant C so that for any origin-symmetric convex body K in $\mathbb{R}^n$

$$|K|^{\frac{n-1}{n}} \leq C \max_{\xi \in S^{n-1}} |K \cap \xi^{\perp}| \quad ?$$

The best-to-date estimate $C \sim n^{1/4}$ is due to Klartag, who removed the logarithmic term from an earlier estimate of Bourgain.

Here $\xi^{\perp}$ is the central hyperplane perpendicular to $\xi \in S^{n-1}$, and $|K|$ is volume of proper dimension.

Hyperplane problem

Hyperplane problem: Does there exist an absolute constant C so that for any origin-symmetric convex body K in $\mathbb{R}^n$

$$|K|^{\frac{n-1}{n}} \leq C \max_{\xi \in S^{n-1}} |K \cap \xi^{\perp}| \quad ?$$

The best-to-date estimate $C \sim n^{1/4}$ is due to Klartag, who removed the logarithmic term from an earlier estimate of Bourgain.

Here $\xi^{\perp}$ is the central hyperplane perpendicular to $\xi \in S^{n-1}$, and $|K|$ is volume of proper dimension.

For certain classes of bodies the question has been answered in affirmative.

- unconditional convex bodies (Bourgain),
- unit balls of subspaces of L_p (Ball, Junge, E.Milman), $C \sim \sqrt{p}, \ p \to \infty$
- intersection bodies (immediate from the Busemann-Petty problem),
- zonoids, duals of bodies with bounded volume ratio (V.Milman-Pajor),
- the Schatten classes (König, Meyer, Pajor)
- for details and other results see the book "Geometry of isotropic log concave measures" by Brazitikos, Giannopoulos, Valettas and Vritsiou.

Hyperplane inequality with arbitrary measure

Let f be an even continuous function on $\mathbb{R}^n$ and let μ be the measure with density f, i.e. $\mu(B) = \int_B f$ for every compact set $B \subset \mathbb{R}^n$.

Hyperplane inequality with arbitrary measure

Let f be an even continuous function on $\mathbb{R}^n$ and let μ be the measure with density f, i.e. $\mu(B) = \int_B f$ for every compact set $B \subset \mathbb{R}^n$.

Does there exist a constant C such that for any origin-symmetric convex body K and any measure μ with even continuous density

$$\mu(K) \leq C \max_{\xi \in S^{n-1}} \mu(K \cap \xi^\perp) \ |K|^{1/n}.$$

A closed bounded set K in $\mathbb{R}^n$ is called a *star body* if every straight line passing through the origin crosses the boundary of K at exactly two points different from the origin, the origin is an interior point of K, and the *Minkowski functional* of K defined by $\|x\|_K = \min\{a \geq 0 : \ x \in aK\}$ is a continuous function on $\mathbb{R}^n$.

Intersection bodies

A closed bounded set K in $\mathbb{R}^n$ is called a *star body* if every straight line passing through the origin crosses the boundary of K at exactly two points different from the origin, the origin is an interior point of K, and the *Minkowski functional* of K defined by $\|x\|_K = \min\{a \geq 0 : \ x \in aK\}$ is a continuous function on $\mathbb{R}^n$.

The *radial function* of a star body K is defined by $r_K(x) = \|x\|_K^{-1}$, $x \in \mathbb{R}^n$. If $x \in S^{n-1}$ then $r_K(x)$ is the radius of K in the direction of x.

A closed bounded set K in $\mathbb{R}^n$ is called a *star body* if every straight line passing through the origin crosses the boundary of K at exactly two points different from the origin, the origin is an interior point of K, and the *Minkowski functional* of K defined by $\|x\|_K = \min\{a \geq 0 : \ x \in aK\}$ is a continuous function on $\mathbb{R}^n$.

The *radial function* of a star body K is defined by $r_K(x) = \|x\|_K^{-1}$, $x \in \mathbb{R}^n$. If $x \in S^{n-1}$ then $r_K(x)$ is the radius of K in the direction of x.

The class of intersection bodies was introduced by Lutwak. We say that a star body K in $\mathbb{R}^n$ is the intersection body of another star body L for every $\xi \in S^{n-1}$,

$$r_K(\xi) = \|\xi\|_K^{-1} = |L \cap \xi^{\perp}|.$$

Intersection bodies

A closed bounded set K in $\mathbb{R}^n$ is called a *star body* if every straight line passing through the origin crosses the boundary of K at exactly two points different from the origin, the origin is an interior point of K, and the *Minkowski functional* of K defined by $\|x\|_K = \min\{a \geq 0 : \; x \in aK\}$ is a continuous function on $\mathbb{R}^n$.

The *radial function* of a star body K is defined by $r_K(x) = \|x\|_K^{-1}$, $x \in \mathbb{R}^n$. If $x \in S^{n-1}$ then $r_K(x)$ is the radius of K in the direction of x.

The class of intersection bodies was introduced by Lutwak. We say that a star body K in $\mathbb{R}^n$ is the intersection body of another star body L for every $\xi \in S^{n-1}$,

$$r_K(\xi) = \|\xi\|_K^{-1} = |L \cap \xi^\perp|.$$

The class of **intersection bodies** is the closure of the class of intersection bodies of star bodies in the radial metric

$$\rho(K,L) = \sup_{\xi \in S^{n-1}} |r_K(\xi) - r_L(\xi)|.$$

A closed bounded set K in $\mathbb{R}^n$ is called a *star body* if every straight line passing through the origin crosses the boundary of K at exactly two points different from the origin, the origin is an interior point of K, and the *Minkowski functional* of K defined by $\|x\|_K = \min\{a \geq 0 : \ x \in aK\}$ is a continuous function on $\mathbb{R}^n$.

The *radial function* of a star body K is defined by $r_K(x) = \|x\|_K^{-1}$, $x \in \mathbb{R}^n$. If $x \in S^{n-1}$ then $r_K(x)$ is the radius of K in the direction of x.

The class of intersection bodies was introduced by Lutwak. We say that a star body K in $\mathbb{R}^n$ is the intersection body of another star body L for every $\xi \in S^{n-1}$,

$$r_K(\xi) = \|\xi\|_K^{-1} = |L \cap \xi^\perp|.$$

The class of **intersection bodies** is the closure of the class of intersection bodies of star bodies in the radial metric

$$\rho(K, L) = \sup_{\xi \in S^{n-1}} |r_K(\xi) - r_L(\xi)|.$$

K.(1998): An origin-symmetric star body K in $\mathbb{R}^n$ is an intersection body if and only if the Fourier transform of $\|x\|_K^{-1}$ is a positive distribution.

A closed bounded set K in $\mathbb{R}^n$ is called a *star body* if every straight line passing through the origin crosses the boundary of K at exactly two points different from the origin, the origin is an interior point of K, and the *Minkowski functional* of K defined by $\|x\|_K = \min\{a \geq 0 : \ x \in aK\}$ is a continuous function on $\mathbb{R}^n$.

The *radial function* of a star body K is defined by $r_K(x) = \|x\|_K^{-1}$, $x \in \mathbb{R}^n$. If $x \in S^{n-1}$ then $r_K(x)$ is the radius of K in the direction of x.

The class of intersection bodies was introduced by Lutwak. We say that a star body K in $\mathbb{R}^n$ is the intersection body of another star body L for every $\xi \in S^{n-1}$,

$$r_K(\xi) = \|\xi\|_K^{-1} = |L \cap \xi^{\perp}|.$$

The class of **intersection bodies** is the closure of the class of intersection bodies of star bodies in the radial metric

$$\rho(K, L) = \sup_{\xi \in S^{n-1}} |r_K(\xi) - r_L(\xi)|.$$

K.(1998): An origin-symmetric star body K in $\mathbb{R}^n$ is an intersection body if and only if the Fourier transform of $\|x\|_K^{-1}$ is a positive distribution.

K.(1998): The unit ball of any finite dimensional subspace of L_p, $p \in (0, 2)$ is an intersection body.

Lutwak's connection: if K is an intersection body and L any origin-symmetric star body in $\mathbb{R}^n$ so that $|K \cap \xi^\perp| \leq |L \cap \xi^\perp|, \forall \xi \in S^{n-1}$, then $|K| \leq |L|$.

Lutwak's connection: if K is an intersection body and L any origin-symmetric star body in $\mathbb{R}^n$ so that $|K \cap \xi^\perp| \le |L \cap \xi^\perp|$, $\forall \xi \in S^{n-1}$, then $|K| \le |L|$.

If K is an intersection body

$$|K|^{\frac{n-1}{n}} \le c_n \max_{\xi \in S^{n-1}} |K \cap \xi^\perp|, \qquad \text{where} \qquad c_n := |B_2^n|^{\frac{n-1}{n}} / |B_2^{n-1}|$$

and B_2^n is the unit Euclidean ball. Note that $c_n \in (\frac{1}{\sqrt{e}}, 1)$.

Lutwak's connection: if K is an intersection body and L any origin-symmetric star body in $\mathbb{R}^n$ so that $|K\cap\xi^\perp| \le |L\cap\xi^\perp|$, $\forall\xi\in S^{n-1}$, then $|K|\le|L|$.

If K is an intersection body

$$|K|^{\frac{n-1}{n}} \le c_n \max_{\xi\in S^{n-1}} |K\cap\xi^\perp|, \qquad \text{where} \qquad c_n := |B_2^n|^{\frac{n-1}{n}}/|B_2^{n-1}|$$

and B_2^n is the unit Euclidean ball. Note that $c_n \in (\frac{1}{\sqrt{e}},1)$.

Put $L=B_2^n$ in Lutwak's connection and assume that $|K|=|B_2^n|$. Then

$$\max_{\xi\in S^{n-1}} |K\cap\xi^\perp| \ge |B_2^n\cap\xi^\perp| = |B_2^{n-1}|.$$

Now divide both sides by equal numbers $|K|^{\frac{n-1}{n}}|$ and $|B_2^n|^{\frac{n-1}{n}}$.

Lutwak's connection: if K is an intersection body and L any origin-symmetric star body in $\mathbb{R}^n$ so that $|K\cap\xi^\perp| \le |L\cap\xi^\perp|$, $\forall \xi\in S^{n-1}$, then $|K|\le|L|$.

If K is an intersection body

$$|K|^{\frac{n-1}{n}} \le c_n \max_{\xi\in S^{n-1}} |K\cap\xi^\perp|, \qquad \text{where} \qquad c_n := |B_2^n|^{\frac{n-1}{n}}/|B_2^{n-1}|$$

and B_2^n is the unit Euclidean ball. Note that $c_n\in(\frac{1}{\sqrt{e}},1)$.

Put $L=B_2^n$ in Lutwak's connection and assume that $|K|=|B_2^n|$. Then

$$\max_{\xi\in S^{n-1}} |K\cap\xi^\perp| \ge |B_2^n\cap\xi^\perp| = |B_2^{n-1}|.$$

Now divide both sides by equal numbers $|K|^{\frac{n-1}{n}}|$ and $|B_2^n|^{\frac{n-1}{n}}$.

It was proved in K.(2012) that the latter inequality holds for arbitrary measures in place of volume as follows. Suppose that K is an intersection body in $\mathbb{R}^n$, and μ is a measure on $\mathbb{R}^n$ with even continuous density. Then

$$\mu(K) \le \frac{n}{n-1} c_n \max_{\xi\in S^{n-1}} \mu(K\cap\xi^\perp)\ |K|^{1/n}.$$

The constant is sharp.

$\sqrt{n}$ estimate

Let L be an origin-symmetric convex body in $\mathbb{R}^n$, and let μ be a measure with even continuous non-negative density f on L. Then

$$\mu(L) \ \leq \ \sqrt{n}\frac{n}{n-1}c_n \max_{\xi\in S^{n-1}} \mu(L\cap\xi^\perp)\ |L|^{1/n}.$$

Recall that $c_n < 1$.

A convex body K in $\mathbb{R}^n$ is called unconditional if there exists a basis $e_i,\ 1 \leq i \leq n$, in $\mathbb{R}^n$ such that for every choice of real numbers x_i and $\delta_i = \pm 1,\ 1 \leq i \leq n$ we have

$$\|\sum_{i=1}^{n} \delta_i x_i e_i\|_K = \|\sum_{i=1}^{n} x_i e_i\|_K.$$

Here $\|x\|_K = \min\{a \geq 0 : \ x \in aK\}$.

A convex body K in $\mathbb{R}^n$ is called unconditional if there exists a basis $e_i,\ 1 \le i \le n$, in $\mathbb{R}^n$ such that for every choice of real numbers x_i and $\delta_i = \pm 1,\ 1 \le i \le n$ we have

$$\|\sum_{i=1}^{n} \delta_i x_i e_i\|_K = \|\sum_{i=1}^{n} x_i e_i\|_K.$$

Here $\|x\|_K = \min\{a \ge 0 : \ x \in aK\}$.

There exists a constant $C > 0$ such that for every $n \in \mathbb{N}$, every unconditional convex body K in $\mathbb{R}^n$ and every measure μ with even continuous non-negative density on K

$$\mu(K) \ \le \ C \max_{\xi \in S^{n-1}} \mu(K \cap \xi^{\perp}) \ |K|^{1/n}.$$

One can take $C = 2e$.

Unit balls of subspaces of L_p, $p \in (-\infty, \infty)$

For $p \in [0,2]$ the unit balls of finite dimensional subspaces of L_p are intersection bodies; K.(1998).

Unit balls of subspaces of $L_p,\ p \in (-\infty, \infty)$

For $p \in [0,2]$ the unit balls of finite dimensional subspaces of L_p are intersection bodies; K.(1998).

For $p > 2$

$$\mu(K) \le n^{1/2-1/p} \frac{n}{n-1} c_n \max_{\xi \in S^{n-1}} \mu(K \cap \xi^\perp)\ |K|^{1/n}.$$

For $p \in [0,2]$ the unit balls of finite dimensional subspaces of L_p are intersection bodies; K.(1998).

For $p > 2$

$$\mu(K) \le n^{1/2-1/p} \frac{n}{n-1} c_n \max_{\xi \in S^{n-1}} \mu(K \cap \xi^\perp)\ |K|^{1/n}.$$

For any $p > 0$ there exists a constant $C(p)$ such that for any $n \in \mathbb{N},\ n > p$, any convex body K in $\mathbb{R}^n$ that is the unit ball of a normed space embedding in L_{-p} and any measure μ with even continuous non-negative density in $\mathbb{R}^n$,

$$\mu(K) \ \le\ C(p) \max_{\xi \in S^{n-1}} \mu(K \cap \xi^\perp)\ |K|^{1/n}.$$

When $p = k \in \mathbb{N},\ 1 \le k \le n-1$ the latter inequality applies to any convex k-intersection body K in $\mathbb{R}^n$.

The volume ratio of a convex body K in $\mathbb{R}^n$ is defined by

$$\text{v.r.}(K) = \inf_E \left\{ \left(\frac{|K|}{|E|} \right)^{1/n} : \ E \subset K, \ E - \text{ellipsoid} \right\}.$$

Duals of bodies with bounded volume ratio

The volume ratio of a convex body K in $\mathbb{R}^n$ is defined by

$$\text{v.r.}(K) = \inf_E \left\{ \left(\frac{|K|}{|E|} \right)^{1/n} : \ E \subset K, \ E - \text{ellipsoid} \right\}.$$

There exists an absolute constant C such that for every $n \in \mathbb{N}$, every origin-symmetric convex body K in $\mathbb{R}^n$ and every measure μ with even continuous non-negative density on K

$$\mu(K) \ \leq \ C \ \text{v.r.}(K^\circ) \max_{\xi \in S^{n-1}} \mu(K \cap \xi^\perp) \ |K|^{1/n}.$$

Stability Theorem. Let K be an intersection body in $\mathbb{R}^n$, let f be an even continuous function on K, $f \geq 1$ everywhere on K, and let $\varepsilon > 0$. Suppose that

$$\int_{K\cap\xi^\perp} f \ \leq \ |K\cap\xi^\perp| + \varepsilon, \qquad \forall \xi \in S^{n-1},$$

then

$$\int_K f \ \leq \ |K| + \frac{n}{n-1}\ c_n\ |K|^{1/n}\varepsilon.$$

Stability Theorem. Let K be an intersection body in $\mathbb{R}^n$, let f be an even continuous function on K, $f \geq 1$ everywhere on K, and let $\varepsilon > 0$. Suppose that

$$\int_{K\cap\xi^\perp} f \ \leq \ |K\cap\xi^\perp| + \varepsilon, \qquad \forall \xi \in S^{n-1},$$

then

$$\int_K f \ \leq \ |K| + \frac{n}{n-1}\ c_n\ |K|^{1/n}\varepsilon.$$

Define the intersection outer volume ratio of a star body L in $\mathbb{R}^n$ by

$$\mathrm{i.r.}(L) = \inf_K \left\{ \left(\frac{|K|}{|L|}\right)^{1/n} :\ L \subset K,\ K - \text{intersection body} \right\}.$$

Stability Theorem. Let K be an intersection body in $\mathbb{R}^n$, let f be an even continuous function on K, $f \geq 1$ everywhere on K, and let $\varepsilon > 0$. Suppose that

$$\int_{K\cap\xi^\perp} f \ \leq \ |K\cap\xi^\perp| + \varepsilon, \qquad \forall \xi \in S^{n-1},$$

then

$$\int_K f \ \leq \ |K| + \frac{n}{n-1} \ c_n \ |K|^{1/n}\varepsilon.$$

Define the intersection outer volume ratio of a star body L in $\mathbb{R}^n$ by

$$\text{i.r.}(L) = \inf_K \left\{ \left(\frac{|K|}{|L|} \right)^{1/n} : \ L \subset K, \ K - \text{intersection body} \right\}.$$

Corollary. Let L be an origin-symmetric star body in $\mathbb{R}^n$. Then for any measure μ with even continuous density on L we have

$$\mu(L) \leq \text{i.r.}(L) \ \frac{n}{n-1} c_n \max_{\xi\in S^{n-1}} \mu(L\cap\xi^\perp)|L|^{1/n}.$$

Proof of Corollary.

Let K be an intersection body in $\mathbb{R}^n$ such that $L \subset K$ and

$$|K|^{1/n} \leq C\ |L|^{1/n}.$$

Proof of Corollary.

Let K be an intersection body in $\mathbb{R}^n$ such that $L \subset K$ and

$$|K|^{1/n} \leq C\ |L|^{1/n}.$$

Let g be the density of the measure μ, and put $f = \chi_K + g\chi_L$. Clearly, $f \geq 1$ everywhere on K. Put

$$\varepsilon = \max_{\xi \in S^{n-1}} \left(\int_{K \cap \xi^\perp} f - |K \cap \xi^\perp| \right) = \max_{\xi \in S^{n-1}} \int_{L \cap \xi^\perp} g$$

and apply Stability Theorem to f, K, ε. We have

$$\mu(L) = \int_L g = \int_K f - \ |K| \leq \frac{n}{n-1} c_n |K|^{1/n} \max_{\xi \in S^{n-1}} \int_{L \cap \xi^\perp} g$$

$$\leq C\ \frac{n}{n-1} c_n |L|^{1/n} \max_{\xi \in S^{n-1}} \mu(L \cap \xi^\perp).$$

Suppose that L is an origin-symmetric **convex** body in $\mathbb{R}^n$. By John's theorem, there exists an origin-symmetric ellipsoid K such that $\frac{1}{\sqrt{n}}K \subset L \subset K$. The ellipsoid K is an intersection body, so

$$\text{i.r.}(L) \le \sqrt{n}.$$

Suppose that L is an origin-symmetric **convex** body in $\mathbb{R}^n$. By John's theorem, there exists an origin-symmetric ellipsoid K such that $\frac{1}{\sqrt{n}}K \subset L \subset K$. The ellipsoid K is an intersection body, so

$$\mathrm{i.r.}(L) \le \sqrt{n}.$$

By Corollary,

$$\mu(L) \le \sqrt{n}\ \frac{n}{n-1} c_n \max_{\xi \in S^{n-1}} \mu(L \cap \xi^\perp) |L|^{1/n}.$$

Unconditional convex bodies

Suppose that L is an unconditional convex body. By a result of Lozanovskii, there exists a linear operator $T : \mathbb{R}^n \to \mathbb{R}^n$ so that

$$T(B_\infty^n) \subset L \subset nT(B_1^n),$$

where B_p^n is the unit ball of the space ℓ_p^n. Let $K = nT(B_1^n)$. By K.(1998), K is an intersection body in $\mathbb{R}^n$.

Suppose that L is an unconditional convex body. By a result of Lozanovskii, there exists a linear operator $T : \mathbb{R}^n \to \mathbb{R}^n$ so that

$$T(B_\infty^n) \subset L \subset nT(B_1^n),$$

where B_p^n is the unit ball of the space ℓ_p^n. Let $K = nT(B_1^n)$. By K.(1998), K is an intersection body in $\mathbb{R}^n$.

Since $|B_1^n| = 2^n/n!$, we have $|K|^{1/n} \le 2e|\det T|^{1/n}$. On the other hand, $|T(B_\infty^n)| = 2^n|\det T|$, and $T(B_\infty^n) \subset L$, so $|K|^{1/n} \le e\ |L|^{1/n}$. Therefore,

$$\mathrm{i.r.}(L) \le e.$$

Suppose that L is an unconditional convex body. By a result of Lozanovskii, there exists a linear operator $T : \mathbb{R}^n \to \mathbb{R}^n$ so that

$$T(B_\infty^n) \subset L \subset nT(B_1^n),$$

where B_p^n is the unit ball of the space ℓ_p^n. Let $K = nT(B_1^n)$. By K.(1998), K is an intersection body in $\mathbb{R}^n$.

Since $|B_1^n| = 2^n/n!$, we have $|K|^{1/n} \leq 2e|\det T|^{1/n}$. On the other hand, $|T(B_\infty^n)| = 2^n|\det T|$, and $T(B_\infty^n) \subset L$, so $|K|^{1/n} \leq e\ |L|^{1/n}$. Therefore,

$$\text{i.r.}(L) \leq e.$$

By Corollary,

$$\mu(L) \leq e\ \frac{n}{n-1} c_n \max_{\xi \in S^{n-1}} \mu(L \cap \xi^\perp)|L|^{1/n}.$$

An n-dimensional normed space $(\mathbb{R}^n, \|\cdot\|)$ embeds isometrically in $L_p,\ -n < p < \infty,\ p \neq 2k,\ k \in \mathbb{N} \cup \{0\}$ if and only if the Fourier transform of the distribution $\frac{1}{\Gamma(-p/2)}\|\cdot\|^p$ is a positive distribution outside of the origin in $\mathbb{R}^n$ (for $p > 0$ this is a theorem, for $p < 0$ can be considered as the definition).

An n-dimensional normed space $(\mathbb{R}^n, \|\cdot\|)$ embeds isometrically in $L_p,\ -n < p < \infty,\ p \neq 2k,\ k \in \mathbb{N} \cup \{0\}$ if and only if the Fourier transform of the distribution $\frac{1}{\Gamma(-p/2)}\|\cdot\|^p$ is a positive distribution outside of the origin in $\mathbb{R}^n$ (for $p > 0$ this is a theorem, for $p < 0$ can be considered as the definition).

For an integer $k,\ 1 \leq k < n$ and star bodies D, L in $\mathbb{R}^n$, we say that D is the k-intersection body of L if for every $(n-k)$-dimensional subspace H of $\mathbb{R}^n$,

$$|D \cap H^\perp| = |L \cap H|,$$

where $H^\perp$ is the k-dimensional subspace orthogonal to H. The closure in the radial metric of the class of all D's that appear as k-intersection bodies of star bodies defines the class of *k-intersection bodies*.

An n-dimensional normed space $(\mathbb{R}^n, \|\cdot\|)$ embeds isometrically in $L_p,\ -n < p < \infty,\ p \neq 2k,\ k \in \mathbb{N} \cup \{0\}$ if and only if the Fourier transform of the distribution $\frac{1}{\Gamma(-p/2)}\|\cdot\|^p$ is a positive distribution outside of the origin in $\mathbb{R}^n$ (for $p > 0$ this is a theorem, for $p < 0$ can be considered as the definition).

For an integer $k,\ 1 \leq k < n$ and star bodies D, L in $\mathbb{R}^n$, we say that D is the k-intersection body of L if for every $(n-k)$-dimensional subspace H of $\mathbb{R}^n$,

$$|D \cap H^\perp| = |L \cap H|,$$

where $H^\perp$ is the k-dimensional subspace orthogonal to H. The closure in the radial metric of the class of all D's that appear as k-intersection bodies of star bodies defines the class of *k-intersection bodies*.

An origin symmetric star body D is a k-intersection body if and only if the space $(\mathbb{R}^n, \|\cdot\|_K)$ embeds in L_{-k}.

Hyperplane inequalities for subspaces of L_p

For $0 < p \leq 2$ the unit ball of any finite-dimensional subspace of L_p is an intersection body, so the hyperplane inequality for intersection bodies with arbitrary measures applies.

For $0 < p \le 2$ the unit ball of any finite-dimensional subspace of L_p is an intersection body, so the hyperplane inequality for intersection bodies with arbitrary measures applies.

For $p > 2$, using the result of Lewis that $d_{BM}(K, B_2^n) \le n^{1/2-1/p}$ if K is the unit ball of an n-dimensional subspace of L_p, we get

$$\mu(K) \le n^{1/2-1/p} \frac{n}{n-1} c_n \max_{\xi \in S^{n-1}} \mu(K \cap \xi^\perp)\ |K|^{1/n}$$

For $0 < p \leq 2$ the unit ball of any finite-dimensional subspace of L_p is an intersection body, so the hyperplane inequality for intersection bodies with arbitrary measures applies.

For $p > 2$, using the result of Lewis that $d_{BM}(K, B_2^n) \leq n^{1/2-1/p}$ if K is the unit ball of an n-dimensional subspace of L_p, we get

$$\mu(K) \leq n^{1/2-1/p} \frac{n}{n-1} c_n \max_{\xi \in S^{n-1}} \mu(K \cap \xi^\perp) \, |K|^{1/n}$$

Kalton, K. 2005: If $-\infty < p < q < 1$, $p \neq 0$ and $q > 0$ then there exists a constant $C = C(p,q)$ so that, for any n with $-n < p$, if K is the unit ball of an n-dimensional normed space X that embeds into L_p, then there is an n-dimensional subspace Y of L_q whose unit ball L satisfies

$$L \subset K \subset C(p,q)L.$$

For $0 < p \le 2$ the unit ball of any finite-dimensional subspace of L_p is an intersection body, so the hyperplane inequality for intersection bodies with arbitrary measures applies.

For $p > 2$, using the result of Lewis that $d_{BM}(K, B_2^n) \le n^{1/2-1/p}$ if K is the unit ball of an n-dimensional subspace of L_p, we get

$$\mu(K) \le n^{1/2-1/p} \frac{n}{n-1} c_n \max_{\xi \in S^{n-1}} \mu(K \cap \xi^\perp)\ |K|^{1/n}$$

Kalton, K. 2005: If $-\infty < p < q < 1$, $p \neq 0$ and $q > 0$ then there exists a constant $C = C(p,q)$ so that, for any n with $-n < p$, if K is the unit ball of an n-dimensional normed space X that embeds into L_p, then there is an n-dimensional subspace Y of L_q whose unit ball L satisfies

$$L \subset K \subset C(p,q)L.$$

Since L is an intersection body (K., 1998), the same argument as before leads to a hyperplane inequality: For any $p > 0$ there exists a constant $C(p)$ such that for any $n \in \mathbb{N}$, $n > p$, any convex body K in $\mathbb{R}^n$ that is the unit ball of a normed space embedding in L_{-p} and any measure μ with even continuous non-negative density in $\mathbb{R}^n$,

$$\mu(K) \ \le\ C(p) \max_{\xi \in S^{n-1}} \mu(K \cap \xi^\perp)\ |K|^{1/n}.$$

This applies to convex k-intersection bodies when $p = k$.

Stability Theorem. Let K be an intersection body in $\mathbb{R}^n$, let f be an even continuous function on K, $f \geq 1$ everywhere on K, and let $\varepsilon > 0$. Suppose that

$$\int_{K\cap\xi^\perp} f \ \leq \ |K\cap\xi^\perp| + \varepsilon, \qquad \forall \xi \in S^{n-1},$$

then

$$\int_K f \ \leq \ |K| + \frac{n}{n-1}\ c_n\ |K|^{1/n}\varepsilon.$$

Stability Theorem. Let K be an intersection body in $\mathbb{R}^n$, let f be an even continuous function on K, $f \geq 1$ everywhere on K, and let $\varepsilon > 0$. Suppose that

$$\int_{K\cap\xi^\perp} f \;\leq\; |K\cap\xi^\perp| + \varepsilon, \qquad \forall \xi \in S^{n-1},$$

then

$$\int_K f \;\leq\; |K| + \frac{n}{n-1}\; c_n \;|K|^{1/n}\varepsilon.$$

Proof. By K. (1997) and Zvavitch (2005),

$$|K\cap\xi^\perp| = \frac{1}{\pi(n-1)}(\|\cdot\|_K^{-n+1})^\wedge(\xi)$$

and

$$\int_{K\cap\xi^\perp} f = \frac{1}{\pi}\left(|x|_2^{-n+1}\int_0^{\frac{|x|_2}{\|x\|_K}} r^{n-2} f(\frac{rx}{|x|_2})\; dr\right)^\wedge (\xi).$$

Stability Theorem. Let K be an intersection body in $\mathbb{R}^n$, let f be an even continuous function on K, $f \geq 1$ everywhere on K, and let $\varepsilon > 0$. Suppose that

$$\int_{K\cap\xi^\perp} f \ \leq \ |K\cap\xi^\perp| + \varepsilon, \qquad \forall \xi \in S^{n-1},$$

then

$$\int_K f \ \leq \ |K| + \frac{n}{n-1}\ c_n\ |K|^{1/n}\varepsilon.$$

Proof. By K. (1997) and Zvavitch (2005),

$$|K\cap\xi^\perp| = \frac{1}{\pi(n-1)}(\|\cdot\|_K^{-n+1})^\wedge(\xi)$$

and

$$\int_{K\cap\xi^\perp} f = \frac{1}{\pi}\left(|x|_2^{-n+1}\int_0^{\frac{|x|_2}{\|x\|_K}} r^{n-2} f(\frac{rx}{|x|_2})\ dr\right)^\wedge(\xi).$$

$$\frac{1}{\pi}\left(|x|_2^{-n+1}\int_0^{\frac{|x|_2}{\|x\|_K}} r^{n-2} f(\frac{rx}{|x|_2})\ dr\right)^\wedge(\xi) \leq \frac{1}{\pi(n-1)}(\|\cdot\|_K^{-n+1})^\wedge(\xi) + \varepsilon.$$

Since K is an intersection body, $(\|x\|_K^{-1})^\wedge \geq 0$, so

$$\int_{S^{n-1}} (\|x\|_K^{-1})^\wedge(\xi) \left(|x|_2^{-n+1} \int_0^{\frac{|x|_2}{\|x\|_K}} r^{n-2} f(\frac{rx}{|x|_2})\ dr \right)^\wedge (\xi)\ d\xi$$

$$\leq \frac{1}{n-1} \int_{S^{n-1}} (\|x\|_K^{-1})^\wedge(\xi)(\|\cdot\|_K^{-n+1})^\wedge(\xi) d\xi + \pi\varepsilon \int_{S^{n-1}} (\|x\|_K^{-1})^\wedge(\xi) d\xi$$

Since K is an intersection body, $(\|x\|_K^{-1})^\wedge \geq 0$, so

$$\int_{S^{n-1}} (\|x\|_K^{-1})^\wedge(\xi) \left(|x|_2^{-n+1} \int_0^{\frac{|x|_2}{\|x\|_K}} r^{n-2} f(\frac{rx}{|x|_2})\ dr \right)^\wedge (\xi)\ d\xi$$

$$\leq \frac{1}{n-1} \int_{S^{n-1}} (\|x\|_K^{-1})^\wedge(\xi)(\|\cdot\|_K^{-n+1})^\wedge(\xi) d\xi + \pi\varepsilon \int_{S^{n-1}} (\|x\|_K^{-1})^\wedge(\xi) d\xi$$

By Spherical Parseval's formula,

$$\int_{S^{n-1}} \|\theta\|_K^{-1} \left(\int_0^{\|\theta\|_K^{-1}} r^{n-2} f(r\theta)\ dr \right) d\theta$$

$$\leq \frac{1}{n-1} \int_{S^{n-1}} \|\theta\|_K^{-n}\ d\theta + \frac{\pi}{(2\pi)^n} \varepsilon \int_{S^{n-1}} (\|x\|_K^{-1})^\wedge(\xi).$$

Since K is an intersection body, $(\|x\|_K^{-1})^\wedge \geq 0$, so

$$\int_{S^{n-1}} (\|x\|_K^{-1})^\wedge(\xi) \left(|x|_2^{-n+1} \int_0^{\frac{|x|_2}{\|x\|_K}} r^{n-2} f(\frac{rx}{|x|_2})\ dr \right)^\wedge (\xi)\ d\xi$$

$$\leq \frac{1}{n-1} \int_{S^{n-1}} (\|x\|_K^{-1})^\wedge(\xi)(\|\cdot\|_K^{-n+1})^\wedge(\xi) d\xi + \pi\varepsilon \int_{S^{n-1}} (\|x\|_K^{-1})^\wedge(\xi) d\xi$$

By Spherical Parseval's formula,

$$\int_{S^{n-1}} \|\theta\|_K^{-1} \left(\int_0^{\|\theta\|_K^{-1}} r^{n-2} f(r\theta)\ dr \right) d\theta$$

$$\leq \frac{1}{n-1} \int_{S^{n-1}} \|\theta\|_K^{-n}\ d\theta + \frac{\pi}{(2\pi)^n} \varepsilon \int_{S^{n-1}} (\|x\|_K^{-1})^\wedge(\xi).$$

Since $f \geq 1$, we estimate the left-hand side from below:

$$= \int_{S^{n-1}} \left(\int_0^{\|\theta\|_K^{-1}} r^{n-1} f(r\theta)\ dr \right) d\theta + \int_{S^{n-1}} \left(\int_0^{\|\theta\|_K^{-1}} (\|\theta\|_K^{-1} - r) r^{n-2} f(r\theta)\ dr \right) d\theta$$

$$\geq \int_K f + \frac{1}{(n-1)n} \int_{S^{n-1}} \|\theta\|_K^{-n}\ d\theta = \int_K f + \frac{1}{n-1} |K|.$$

We have

$$\int_K f \le |K| + \frac{\pi}{(2\pi)^n}\varepsilon \int_{S^{n-1}} (\|x\|_K^{-1})^\wedge(\xi)d\xi.$$

We have

$$\int_K f \le |K| + \frac{\pi}{(2\pi)^n}\varepsilon \int_{S^{n-1}} (\|x\|_K^{-1})^\wedge(\xi)d\xi.$$

To estimate the second summand, we use the formula for the Fourier transform

$$(|x|_2^{-n+1})^\wedge(\xi) = \frac{2\pi^{\frac{n+1}{2}}}{\Gamma(\frac{n-1}{2})}|\xi|_2^{-1}.$$

By Parseval's formula and Hölder's inequality,

$$\int_{S^{n-1}} (\|x\|_K^{-1})^\wedge(\xi)d\xi = \frac{\Gamma(\frac{n-1}{2})}{2\pi^{\frac{n+1}{2}}} \int_{S^{n-1}} (\|x\|_K^{-1})^\wedge(\xi)(|x|_2^{-n+1})^\wedge(\xi)d\xi$$

$$= \frac{(2\pi)^n\Gamma(\frac{n-1}{2})}{2\pi^{\frac{n+1}{2}}} \int_{S^{n-1}} \|x\|_K^{-1}\ dx$$

$$\le \frac{(2\pi)^n\Gamma(\frac{n-1}{2})|S^{n-1}|^{\frac{n-1}{n}}}{2\pi^{\frac{n+1}{2}}} \left(\int_{S^{n-1}} \|x\|_K^{-n}\ dx\right)^{\frac{1}{n}}$$

$$= \frac{(2\pi)^n\Gamma(\frac{n-1}{2})|S^{n-1}|^{\frac{n-1}{n}}}{2\pi^{\frac{n+1}{2}}} n^{\frac{1}{n}}|K|^{\frac{1}{n}}.$$