Construction of pathological Gâteaux differentiable functions

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Introduction

Definition (The jump property)

Let F be a function between real Banach spaces X and Y. We say that F has the *jump property* if F is Gâteaux differentiable on X and there exists $\alpha > 0$ such that

$$\|F'(x) - F'(y)\| \geq \alpha \text{ whenever } x, y \in X \text{ and } x \neq y.$$

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- (ℓ^p, ℓ^q) has the jump property if, and only if, $p \leq q$.
- The couple $(X, \mathbb{R})$ fails to have the jump property.

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(X, Y) has the jump property, $Y \subset Z \Rightarrow (X, Z)$ has the jump property.

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Let X and Y be Banach spaces. Suppose that there exist:

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- A total, bounded biorthogonal system $\{e_n, e_n^*\}_n \subset X \times X^*$, and
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such that, for each $h \in X$, the series

$$\sum_{n=1}^{\infty} e_n^*(h) f_{2n} \quad \text{and} \quad \sum_{n=1}^{\infty} e_n^*(h) f_{2n-1}$$

converge.

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converge. Then, the couple (X, Y) has the jump property.

Applications

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Proof

- Ovsepian and Pelczysnki: There exists a bounded, total biorthogonal system $\{e_n, e_n^*\}_n \subset X \times X^*$ such that

$$\|e_n\| = 1 \text{ for all } n \in \mathbb{N}, \quad \text{and} \quad X = \overline{\text{span}\{e_n\}_n}.$$

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- $(f_n)_n =$ unit vector basis of c_0.Then,

$$\|\sum_n e_n^*(h) f_{2n}\| \leq \|h\| \quad \text{and} \quad \|\sum_n e_n^*(h) f_{2n-1}\| \leq \|h\|.$$

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A basis $(e_n)_n$ is *subsymmetric* if it is unconditional and $(e_n)_n \simeq (e_{k_n})_n$.

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Let $(e_n^*)_n$ be the sequence of biorthogonal functionals associated to $(e_n)_n$, and set, for each n, $f_n = U(e_n)$.

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$$U(h) = \sum_n e_n^*(h) f_n \quad \text{for all} \quad h \in X.$$

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- Subsymmetry of $(f_n)_n \Rightarrow (f_n)_n \simeq (f_{2n})_n$ and $(f_n)_n \simeq (f_{2n-1})_n$. Thus,

$$\sum_n e_n^*(h) f_{2n} \quad \text{and} \quad \sum_n e_n^*(h) f_{2n-1} \quad \text{converge for all} \quad h \in X.$$

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Proof

Let $P : Y \oplus Y \to Y$ be an isomorphism, and define

$$T(h) = P\left(U(h), 0\right) \quad \text{and} \quad S(h) = P\left(0, U(h)\right), \quad h \in X.$$

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- $(f_n)_n$ is an unconditional basic sequence in Y with $\inf \|f_n\| > 0$.
- And for each $h \in X$,

$$\sum_n e_n^*(h) f_{2n-1} = T(h) \quad \text{and} \quad \sum_n e_n^*(h) f_{2n} = S(h).$$

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$2 \geq p \geq q \geq 1$ & $p \neq 1 \Rightarrow (L^p([0,1]), L^q([0,1]))$ has the jump property.

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- (L^p, L^p) has the jump property (if $p > 1$).

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- If $q \in [1, p]$, then $L^p \subset L^q$.

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Example

If T is the Tsirelson's space, then the couple (T, T) has the jump property. (T does not have any subsymmetric Schauder basis.)

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Proof

- $(e_n)_n =$ canonical basis of T. Then, $(e_n)_n$ is uncondtional.
- $(e_n)_n \simeq (e_{2n-1})_n$ and $(e_n)_n \simeq (e_{2n})_n$. $\Rightarrow T \simeq T \oplus T$.

Lemma

For each norm $|||\cdot|||$ on $\mathbb{R}^2$, each $p = (q, r) \in \mathbb{R}^2$ with $q < r$ and each $\varepsilon > 0$ there is a $\mathcal{C}^1$ function $\varphi = \varphi_{p,\varepsilon} : \mathbb{R}^2 \to \mathbb{R}^2$ so that:

(i) $|||\varphi(s,t)||| \leq \varepsilon$ for all $(s,t) \in \mathbb{R}^2$,

(ii) $\varphi(s,t) = 0$ whenever $s < q$,

(iii) $|||\frac{\partial \varphi}{\partial s}(s,t)||| \leq \varepsilon$ for each $(s,t) \in \mathbb{R}^2$,

(iv) $|||\frac{\partial \varphi}{\partial t}(s,t)||| \leq 1$ for each $(s,t) \in \mathbb{R}^2$, and

(v) $|||\frac{\partial \varphi}{\partial t}(s,t)||| = 1$ whenever $s \geq r$.

Proof

Let $f : \mathbb{R} \to \mathbb{R}$ be the function defined by the expression

$$f(\alpha) = \frac{1}{|||(-\sin\alpha, \cos\alpha)|||}.$$

Then f is Lipschitzian on $\mathbb{R}$ and there exist $C > c > 0$ so that

$$c \leq f(\alpha) \leq C, \quad \text{for all} \quad \alpha \in \mathbb{R}.$$

Thus, there exist a (unique) $\mathcal{C}^1$ function $\alpha : \mathbb{R} \to \mathbb{R}$ with

$$\alpha'(t) = \frac{1}{|||\big(-\sin\alpha(t), \cos(\alpha t)\big)|||} \quad \text{and} \quad \alpha(0) = 0.$$

Let $\gamma : \mathbb{R} \to \mathbb{R}^2$ be the mapping defined by

$$\gamma(t) = (\cos(\alpha(t), \sin(\alpha(t))\,, \quad t \in \mathbb{R}.$$

Let $\beta : \mathbb{R} \to \mathbb{R}$ be a $\mathcal{C}^\infty$ function such that

$$0 \leq \beta(s) \leq 1, \;\; \beta(s) = 0 \;\; \textit{if} \;\; s \leq q \quad \text{and} \quad \beta(s) = 1 \;\; \text{if} \;\; s \geq r.$$

Consider, for n big enough,

$$\varphi(s,t) = n^{-1}\beta(s)\gamma(nt).$$

Construction of F

- Pick $M \geq 1$ with $\|e_n\| \leq M$ and $\|e_n^*\| \leq M$ for all n.
- Fix $\varepsilon \in (0,1)$ and let $\varepsilon_k > 0$ such that $\sum_k \varepsilon_k = \varepsilon/2M$.
- Let us write $\mathbb{P} = \{(q,r) \in \mathbb{Q}^2 : q < r\}$. Let

$$k \mapsto \big(n_k, (q_k, r_k)\big)$$

  be a bijection $\mathbb{N} \to \mathbb{N} \times \mathbb{P}$ such that

$$n_k \neq k, \quad \text{for all} \quad k \in \mathbb{N}.$$

- For each $k \in \mathbb{N}$ and each $(s,t) \in \mathbb{R}^2$ define

$$i_k(s,t) = tf_{2k-1} + sf_{2k}$$

  and

$$\|(s,t)\|_k = \|tf_{2k-1} + sf_{2k}\| = \|i_k(s,t)\|.$$

- $\|\cdot\|_k$ is a norm on $\mathbb{R}^2$.

There exists a $\mathcal{C}^1$ function $\varphi_k : \mathbb{R}^2 \to \mathbb{R}^2$ such that:

$$\|\varphi_k(s,t)\|_k \leq \varepsilon_k \text{ for all } \ (s,t) \in \mathbb{R}^2,$$
$$\varphi_k(s,t) = 0 \ \text{ whenever } \ s < q_k,$$
$$\|\frac{\partial \varphi_k}{\partial s}(s,t)\|_k \leq \varepsilon_k \ \text{ for all } \ (s,t) \in \mathbb{R}^2,$$
$$\|\frac{\partial \varphi_k}{\partial t}(s,t)\|_k \leq 1 \ \text{ for all } \ (s,t) \in \mathbb{R}^2, \text{and}$$
$$\|\frac{\partial \varphi_k}{\partial t}(s,t)\|_k = 1 \ \text{ whenever } \ s \geq r_k.$$

Let us write, for each $x \in X$,

$$F_k(x) = (i_k \circ \varphi_k)\left(e^*_{n_k}(x), e^*_k(x)\right).$$

Then, F_k is a $\mathcal{C}^1$ function, and $\|F_k(x)\| \leq \varepsilon_k$ for all x. The formula

$$F(x) = \sum_k F_k(x)$$

defines a continuous, bounded function $F : X \to Y$.

Gâteaux differentiability of F

Lemma (Criterion of differentiability)

Let $(F_k)_k$ be a sequence of Gâteaux differentiable mappings between X and Y. Suppose that:

1. $\sum_{n=1}^{\infty} F_k$ converges pointwise to a function $F : X \to Y$.
2. For each $h \in X$, the series $\sum_{n=1}^{\infty} F_k'(x)(h)$ converges uniformly with respect to $x \in X$.

Then, F is Gâteaux differentiable on X, and for each $x, h \in X$,

$$F'(x)(h) = \sum_k F_k'(x)(h).$$

Unconditionality of the sequence $(f_n)_n \Rightarrow$ Condition (2). Thus, F is Gâteaux differentiable at every $x \in X$.

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Unconditionality of the sequence $(f_n)_n \Rightarrow$ Condition (2). Thus, F is Gâteaux differentiable at every $x \in X$. Moreover,

$$\sup_{x \in X} \|F'(x)\| < \infty,$$

and F is also Lipschitzian on X.

F has the jump property

- Fix $x, y \in X$ with $x \neq y$. Pick $m \in \mathbb{N}$ with $e_m^*(x) \neq e_m^*(y)$. We can assume that $e_m^*(x) < e_m^*(y)$.
- Let $q, r \in \mathbb{Q}$ such that

$$e_m^*(x) < q < r < e_m^*(y).$$

- Find $k \in \mathbb{N}$ so that $m = n_k$, $q = q_k$ and $r = r_k$.
- $\frac{\partial \varphi_k}{\partial t}(e_{n_k}^*(x), e_k^*(x)) = 0 \Rightarrow F_k'(x)(e_k) = 0.$
- $\|\frac{\partial \varphi_k}{\partial t}(e_{n_k}^*(y), e_k^*(y))\|_k = 1 \Rightarrow \|F_k'(y)(e_k)\| = 1.$
- Therefore,

$$\|F_k'(x)(e_k) - F_k'(y)(e_k)\| = 1.$$

- For each $j \neq k$ we have

$$\|F_j'(x)(e_k)\| \leq \varepsilon_k \quad \text{and} \quad \|F_j'(y)(e_k)\| \leq \varepsilon_k.$$

- Consequently,

$$\begin{aligned} \|F'(x) - F'(y)\| &\geq M^{-1}\|F'(x)(e_k) - F'(y)(e_k)\| \\ &\geq M^{-1}(1 - \sum 2\varepsilon_k) > M^{-1}(1 - \varepsilon). \end{aligned}$$

Open problems

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Under the hypothesis of the theorem, for every $a = (a_n)_n \in \ell^\infty$,

$$L_a(h) = \sum_n e_n^*(h)\left(a_{2n}f_{2n} + a_{2n-1}f_{2n-1}\right), \quad h \in X$$

defines an operator $L_a \in \mathcal{L}(X,Y)$, and there exist $C_1, C_2 > 0$ with

$$C_1\|a\|_\infty \leq \|L_a\| \leq C_2\|a\|_\infty.$$

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Under the hypothesis of the theorem, for every $a = (a_n)_n \in \ell^\infty$,

$$L_a(h) = \sum_n e_n^*(h)\left(a_{2n} f_{2n} + a_{2n-1} f_{2n-1}\right), \quad h \in X$$

defines an operator $L_a \in \mathcal{L}(X, Y)$, and there exist $C_1, C_2 > 0$ with

$$C_1 \|a\|_\infty \le \|L_a\| \le C_2 \|a\|_\infty.$$

In particular, $\ell^\infty \subset \mathcal{L}(X, Y)$.

Open problems

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Problem

- If (X, Y) has the jump property, do we have $\ell^\infty \subset \mathcal{L}(X, Y)$?
- Does the couple $(JT, \mathbb{R}^2)$ have the jump property? ($\mathcal{L}(JT, \mathbb{R}^2)$ does not contain isomorphic copies of ℓ^∞).

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