Approximating the covariance matrix with heavy tailed columns and RIP.

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based on a joint work with

O. Guédon, A. Pajor and N. Tomczak-Jaegermann

(the paper “On the interval ...” available at: http://www.math.ualberta.ca/~alexandr/)

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Notations

\langle \cdot, \cdot \rangle \text{ denotes the canonical inner product on } \mathbb{R}^n.

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A random vector \( X \in \mathbb{R}^n \) is called isotropic if for all \( y \in \mathbb{R}^n \).

\[ \mathbb{E} \langle X, y \rangle = 0 \quad \text{and} \quad \mathbb{E} |\langle X, y \rangle|^2 = |y|^2. \]

In other words, if \( X \) is centered and its covariance matrix is the identity:

\[ \mathbb{E} X \otimes X = Id \]

(recall \( (X \otimes Y)(z) = \langle X, z \rangle Y \) or \( X \otimes Y = \{Y_iX_j\}_{ij} \)).
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For an \( n \times N \) matrix \( T \) its operator norm from \( \ell_2^N \) to \( \ell_2^n \) is denoted by

\[ \|T\| = \sup_{|x|=1} |Tx|. \]
KLS problem

We consider the following model: \( X_1, \ldots, X_N \) are independent random vectors in \( \mathbb{R}^n \). For simplicity we assume that they are identically distributed and isotropic.
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Approximation of covariance matrix
(Kannan-Lovász-Simonovits (KLS) question):

How many random vectors \( X_i \) are needed for the empirical covariance matrix

\[
\frac{1}{N} \sum_{i=1}^{N} X_i \otimes X_i
\]

to approximate the identity with overwhelming probability?

(In Asymptotic Geometric Analysis this question was first asked about vectors uniformly distributed in an isotropic convex body. The approximation was needed in order to estimate the complexity of an algorithm computing the volume of the body).
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or, equivalently,

$$\forall y \in S^{n-1} \quad 1 - \varepsilon \leq \frac{1}{N} \sum_{i=1}^{N} \langle X_i, y \rangle^2 \leq 1 + \varepsilon.$$
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Aubrun (07): $N \sim n/\varepsilon^2$ if $X_1$ is unconditional with $\text{Prob} \geq 1 - \exp(-cn^{1/5})$. 
Solution of KLS Problem in log-concave setting.

Theorem (Adamczak-LPT, 2010)

Let $X_1,\ldots,X_N$ be independent isotropic log-concave random vectors. Let $\varepsilon \in (0,1)$. Then for $N \geq Cn/\varepsilon^2$ one has

$$\mathbb{P} \left( \sup_{y \in S^{n-1}} \left| \frac{1}{N} \sum_{i=1}^{N} (\|X_i, y\|^2 - \mathbb{E}\|X_i, y\|^2) \right| \leq \varepsilon \right) \geq 1 - \exp \left( -c \sqrt{n} \right).$$

Remark. A measure $\mu$ on $\mathbb{R}^n$ is log-concave if for every measurable $A, B \subset \mathbb{R}^n$ and every $\theta \in [0,1]$,

$$\mu(\theta A + (1-\theta) B) \geq \theta \mu(A)(1-\theta).$$
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Relations to standard Random Matrix Theory (RMT)

RMT studies in particular limit behavior of singular numbers of random matrices. Recall for $n \times N$ matrix $A$, the largest and the smallest singular values are defined as

$$s_1(A) = \sup_{|x|=1} \|Ax\| = \|A\| \quad \text{and} \quad s_n(A) = \inf_{|x|=1} \|Ax\| = 1/\|A^{-1}\|.$$

Classical result is the Bai-Yin Theorem.

**Theorem (Bai-Yin)**

*Let $A$ be an $n \times N$ random matrix with i.i.d. entries whose 4-th moments are bounded. Let

$$\beta = \lim_{n \to \infty} \frac{n}{N} \in (0, 1).$$

Then

$$1 - \sqrt{\beta} = \lim_{n \to \infty} s_n(A/\sqrt{N}) \leq \lim_{n \to \infty} s_1(A/\sqrt{N}) = 1 + \sqrt{\beta}.$$*
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**AGA point of view:** We are interested in **asymptotic non-limit** behavior, i.e. we would like to provide the quantitative estimates on the rate of convergence.
Theorem (ALPT)

Let \( n \leq N \). Let \( A \) be a random \( n \times N \) matrix, whose columns \( X_1, \ldots, X_N \) are isotropic log-concave independent random vectors in \( \mathbb{R}^n \). Denoting \( \beta = n/N \) we have

\[
1 - C \sqrt{\beta} \leq s_n(A/\sqrt{N}) \leq s_1(A/\sqrt{N}) \leq 1 + C \sqrt{\beta},
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with probability at least \( 1 - 2 \exp(-c\sqrt{n}) \).
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Compare with the **Bai-Yin** Theorem:

\[
1 - \sqrt{\beta} = \lim_{n \to \infty} s_n \leq \lim_{n \to \infty} s_1 = 1 + \sqrt{\beta}.
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Question: Under what conditions can the KLS problem be solved with $N \sim n$?

For example, is it enough to assume that $|X_i| \leq C \sqrt{n}$ with high probability and

$\sigma_q(X_1) := \sup_{|x| = 1} |\langle X_1, x \rangle|^q \leq C$ for some $q > 2$?

Vershynin (2012): For $q > 4$ if $\sigma_q(X_1) \leq C_1$ and if $|X_i| < C_1 \sqrt{n}$ a.s. then

$\|N \sum_{i=1}^{N} X_i \otimes X_i - I\| \leq C (\ln \ln n)^{2(n/N)^{1/2} - 1/q}$.

He also conjectured that "$\ln \ln n$ is not needed for an appropriate $q$, probably $q = 4$ or even $q > 2$."

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Srivastava and Vershynin (2013): A solution (in average) under strong assumption on projections: there is $\eta > 0$ such that for every projection of rank $k$ and every $t \geq C\sqrt{k}$,

$$\mathbb{P} (|PX_1| \geq t) \leq C/t^{2(1+\eta)}$$
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**Mendelson and Paouris (2012, 2014):** A solution with high probability

1. For $q > 4$ assuming that $X_1$ is unconditional and that for some $p > 2$

   $$\exists p > 2 : \|X_1\|_{\ell_p^n} \leq Cn^{1/p} \text{ a.s.}$$
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In both MP and SV works: for i.i.d. entries with bounded moment $q > 4$. 
Theorem (GLPT)

Let $X_1, \ldots, X_N$ be independent isotropic random vectors. Let $4 < q \leq 8$ and $p < q - 4$. Assume that

$$\forall y \in S^{n-1} \quad \forall t > 0 \quad \mathbb{P} \left( |\langle X, y \rangle| > t \right) \leq t^{-q}.$$ 

Then with high probability

$$\left\| \frac{1}{N} \sum_{i=1}^{N} X_i \otimes X_i - \text{Id} \right\| \leq \frac{C}{N} \max_{i \leq N} |X_i|^2 + C(p, q) \left( \frac{n}{N} \right)^{p/q}.$$
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In particular,

if \( \max_{i \leq N} |X_i|^2 \leq n^{p/q} N^{1-p/q} \) then \( \left\| \frac{1}{N} \sum_{i=1}^{N} X_i \otimes X_i - \operatorname{Id} \right\| \leq C(p, q) \left( \frac{n}{N} \right)^{p/q} \).
Candes and Tao (2005) introduced the concept of Restricted Isometry Property (RIP) for a given matrix $T$ in search for sufficient conditions for $T$ to satisfy some “reconstruction” conditions from “compressed sensing” and coding theory.
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**RIP parameter of order $m$**

is the smallest number $\delta = \delta_m(T)$ such that for every $m$-sparse vector $x \in \mathbb{R}^N$

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($x$ is $m$-sparse if it has at most $m$ non-zero coordinates).
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There are many papers on these topics.
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**basis pursuit algorithm (exact reconstruction by $\ell_1$ minimization)**

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Donoho (2005) showed that the later condition is equivalent to a condition on the neighborliness of polytopes in $\mathbb{R}^n$ (i.e. the above is equivalent to the following: $TB_1^N$ is $m$-centrally-neighborly, that is every set of $m$ vertices containing no opposite pairs forms a vertex set of a face).
Let $X_1, ..., X_N$ be i.i.d. random vectors in $\mathbb{R}^n$ and assume that their Euclidean norms are concentrated around $\sqrt{n}$. Let $T$ be an $n \times N$ matrix whose columns are $X_i/\sqrt{n}$. For example, for isotropic log-concave vectors. Question. Under what (weakest) conditions on $X_i$'s can one obtain RIP?
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In the Gaussian case, the Bernoulli ($\pm 1$) case, the sub-Gaussian case one has $\delta_{2m} \leq \delta_0$ with high probability for

$$m = \frac{Cn}{\ln(2N/n)}.$$

Many works by: Baraniuk, Candes, Cohen, Dahmen, Davenport, DeVore, Donoho, Kashin, Mendelson, Pajor, Romberg, Rudelson, Tao, Temlyakov, Vershynin, Tomczak-Jaegermann, Wakin...
Let $X_1, \ldots, X_N$ be i.i.d. random vectors in $\mathbb{R}^n$ and assume that their Euclidean norms are concentrated around $\sqrt{n}$. Let $T$ be an $n \times N$ matrix whose columns are $X_i/\sqrt{n}$.

In the Gaussian case, the Bernoulli ($\pm 1$) case, the sub-Gaussian case one has $\delta_{2m} \leq \delta_0$ with high probability for

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**ALPT (2011):** Similar estimates with $m = Cn/\ln^2(2N/n)$ provided that

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For example, for isotropic log-concave vectors.
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**Question.** Under what (weakest) conditions on $X_i$’s can one obtain RIP?
Main Results

Theorem (GLPT)

Let $q > 4$ and $p > \frac{4}{q-4}$. Let $X_1, ..., X_N$ be independent random vectors in $\mathbb{R}^n$ such that their Euclidean norms are concentrated around $\sqrt{n}$ and assume

$$\forall y \in S^{n-1} \quad \mathbb{P} (|\langle X_i, y \rangle| > t) \leq C/t^q.$$ 

Then the matrix $T$ whose columns are $X_i/\sqrt{n}$ satisfies $\delta_{2m}(T) \leq \delta_0$ with high probability for

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Sharpness: For $p < \frac{2}{q-2}$, one can’t get better than

$$C_0(p, q) \frac{n}{(N/n)^p}.$$
Main Results

Theorem (GLPT)

Let $\alpha \in (0, 2]$. Let $X_1, \ldots, X_N$ be independent random vectors in $\mathbb{R}^n$ such that their Euclidean norms are concentrated around $\sqrt{n}$ and assume

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Then the matrix $T$ whose columns are $X_i/\sqrt{n}$ satisfies $\delta_{2m}(T) \leq \delta_0$ with high probability for

$$m = \frac{C_\alpha n}{(\ln(2N/n))^{2/\alpha}}.$$ 

Sharpness: The bound on $m$ is sharp up to constant $C_\alpha$. 

Ideas of proofs

The main technical tool is obtaining bounds on the following two parameters.

1. \( A_m := \sup_{a \in S^{N-1}} |\text{supp}(a)| \leq m \left| \sum_{i=1}^N a_i X_i \right| \).

   Note, if \( A \) is the matrix with columns \( X_i \), then \( A_m \) is the supremum of norms of submatrices consisting of \( m \) columns of \( A \).

   The problem of estimating \( A_m \) is interesting by itself, although for KLS problem only \( m = n \) is needed.

2. \( B_m := \sup_{a \in S^{N-1}} |\text{supp}(a)| \leq m \left| \sum_{i=1}^N a_i X_i \right|^2 - \left| \sum_{i=1}^N a_i X_i \right|^2 \).

   \( B_m \) is related to concentration. An upper bound on it plays the crucial role for RIP.
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For $m \leq N$ and random vectors $X_1, ..., X_N$ in $\mathbb{R}^n$, define $A_m$ and $B_m$ by

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RIP parameter $\delta_m$ can be rewritten as

$$\delta_m(A/\sqrt{n}) = \sup_{a \in S^{N-1}} \left| \frac{1}{n} |Aa|^2 - 1 \right|$$
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$$\leq \frac{1}{n} \sup_{\substack{a \in S^{N-1} \mid \text{supp}(a) \leq m}} \left| \frac{1}{n} |Aa|^2 - \frac{1}{n} \sum_{i=1}^N a_i^2 |X_i|^2 \right| + \sup_{\substack{a \in S^{N-1} \mid \text{supp}(a) \leq m}} \left| \frac{1}{n} \sum_{i=1}^N a_i^2 |X_i|^2 - 1 \right|$$
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$$\leq \frac{1}{n} B_m^2 + \sup_{a \in S^{N-1}} \left| \sum_{i=1}^{N} a_i^2 \left( \frac{1}{n} |X_i|^2 - 1 \right) \right|$$
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Note that,

$$\max_{i \leq N} \left| \frac{1}{n} |X_i|^2 - 1 \right| = \delta_1(A/\sqrt{n}) \leq \delta_m(A/\sqrt{n}),$$

that is, concentration of $|X_i|$ around $\sqrt{n}$ is needed.
We need to estimate

$$\mathbb{P} \left( \sup_{a \in S^{n-1}} \left| \sum_{i=1}^{N} \left( \langle X_i, a \rangle^2 - \mathbb{E} \langle X_i, a \rangle^2 \right) \right| > t \right).$$
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First, using symmetrization we pass to

$$\mathbb{P}\left(\sup_{a \in S^{n-1}} \left| \sum_{i=1}^{N} \varepsilon_i \langle X_i, a \rangle^2 \right| > t \right),$$

where $\varepsilon_i$ are independent Bernoulli $\pm 1$ random variables.
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where \( \varepsilon_i \) are independent Bernoulli \( \pm 1 \) random variables.

Conditioning on \( X_i \) and considering decreasing rearrangement,

\[ \left| \sum_{i=1}^{N} \varepsilon_i \langle X_i, a \rangle^2 \right| \leq \sum_{i=1}^{m} \langle X_i, a \rangle^2 * 2 + \sum_{i=m+1}^{N} \varepsilon_{\pi(i)} \langle X_i, a \rangle^2 * 2 , \]

for some permutation \( \pi \).
Now,

\[
\sum_{i=1}^{m} \langle X_i, a \rangle^* 2^2 \leq A_m^2
\]

and using Hoeffding’s inequality, for every \( t > 0 \)

\[
\mathbb{P}(\varepsilon_i) \left( \left| \sum_{i=m+1}^{N} \varepsilon_{\pi(i)} \langle X_i, a \rangle^* 4 \right| \geq t \sqrt{\sum_{i=m+1}^{N} \langle X_i, a \rangle^* 4} \right) \leq 2 \exp\left(-\frac{t^2}{2}\right).
\]
Now,
\[ \sum_{i=1}^{m} \langle X_i, a \rangle^*^2 \leq A_m^2 \]
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Finally we estimate
\[ \mathbb{P} \left( \sum_{i=m+1}^{N} \langle X_i, a \rangle^*^4 > s \right) \]
and choose parameters appropriately (\( m = n, t = \sqrt{n}, ... \)).
Using decoupling argument,

\[ \left| \sum_{i \neq j} \langle a_i X_i, a_j X_j \rangle \right| = 2^{2-N} \left| \sum_{I \subset \{1, 2, \ldots, N\}} \langle \sum_{i \in I} a_i X_i, \sum_{j \in I^c} a_j X_j \rangle \right|. \]
Bounds on $A_m, B_m$

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We denote

$$Q(a, I, I^c) := \left| \sum_{i \in I} a_i X_i, \sum_{j \in I^c} a_j X_j \right|$$

and

$$Q_m(I) = \sup_{a \in \mathbb{S}^{N-1}} Q(a, I, I^c).$$
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\]

Therefore

\[
B_m^2 = \sup_{a \in S^{N-1} \mid |\text{supp}(a)| \leq m} \left| \sum_{i \neq j} \langle a_i X_i, a_j X_j \rangle \right| \leq 2^{2-N} \sum_{I \subseteq \{1,2,\ldots,N\}} Q_m(I).
\]
We prove that for some $\gamma \in [1/2, 1)$ and every $\epsilon \in (0, 1)$, $t > 1$, with high probability
\[ Q_m(I) \leq (1 + \epsilon)(Q_\gamma m(I) + tA_m). \]
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The we use iteration procedure choosing appropriate $\varepsilon$ and $t$ on every step and controlling probability. It will give a bound of the type

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