Davis-Garsia Inequalities for Hardy Martingales

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Topics

1. Basic Examples

2. Maximal Functions

3. Davis Decomposition

4. Martingale Transforms and Consequences

5. Davis Garsia Inequalities

The main sources

A. Pelczynski, Banach Spaces of analytic functions and absolutely summing operators, (1977)

J. Bourgain. *Embedding L^1 to L^1/H^1*, TAMS 278 (1983).

PFXM. A decomposition for Hardy Martingales, Indiana Univ. Math. J. (2012)

PFXM. A decomposition for Hardy Martingales II, Math. Proc. Cambr. Philos. Soc. (2014)

Complex analytic Hardy Spaces

$f \in L^p(\mathbb{T}, X)$, $\mathbb{T} = \{e^{i\theta} : |\theta| \le \pi\}$, $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$.

The harmonic extension of f to the unit disk

$$f(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1 - |z|^2}{|z - e^{i\alpha}|^2} f(e^{i\alpha}) d\alpha, \qquad z \in \mathbb{D}.$$

Define $f \in H^p(\mathbb{T}, X)$ if $f \in L^p(\mathbb{T}, X)$ and the **harmonic extension of** f **is analytic** in $\mathbb{D}$.

Hardy Martingales $H^1(\mathbb{T}^{\mathbb{N}}, X)$

$\mathbb{T}^{\mathbb{N}}$ the infinite torus-product with Haar measure $d\mathbb{P}$.

$F_k : \mathbb{T}^{\mathbb{N}} \to \mathbb{C}$ is $\mathcal{F}_k$ measurable iff

$$F_k(x) = F_k(x_1, \ldots, x_k), \quad x = (x_i)_{i=1}^{\infty}$$

An $(\mathcal{F}_k)$ martingale $F = (F_k)$ with differences $\Delta F_k = F_k - F_{k-1}$ is a **Hardy martingale** if

$$y \to \Delta F_k(x_1, \ldots, x_{k-1}, y) \quad \in H_0^1(\mathbb{T}, X).$$

Conditional expectation $\mathbb{E}_k F$ is integration

$$\mathbb{E}_k F(x) = \int_{\mathbb{T}^{\mathbb{N}}} F(x_1, \ldots, x_k, w) d\mathbb{P}(w).$$

Example: Maurey's embedding.

Fix $\epsilon > 0, w = (w_k) \in \mathbb{T}^{\mathbb{N}}$. Put $\varphi_1(w) = \epsilon w_1$, and

$$\varphi_n(w) = \varphi_{n-1}(w) + \epsilon(1 - |\varphi_{n-1}(w)|)^2 w_n.$$

Then $\lim |\varphi_n| = 1$ and $\varphi = \lim \varphi_n$ is uniformly distributed over $\mathbb{T}$.

For any $f \in H^1(\mathbb{T}, X)$

$$F_n(w) = f(\varphi_n(w)), \qquad w \in \mathbb{T}^{\mathbb{N}}$$

is an integrable Hardy martingale with **uniformly small** increments

$$\sup_{n\in\mathbb{N}} \mathbb{E}(\|F_n\|_X) = \int_{\mathbb{T}} \|f\|_X dm \quad \text{and} \quad \|\Delta F_n\|_X \leq 2\epsilon \int_{\mathbb{T}} \|f\|_X dm.$$

Pointwise estimates for ΔF_n.

Fix $w \in \mathbb{T}^{\mathbb{N}}, \quad n \in \mathbb{N}, \quad z = \varphi_n(w), \quad u = \varphi_{n-1}(w)$

$$\Delta F_n(w) = f(\varphi_n(w)) - f(\varphi_{n-1}(w)).$$

Cauchy integral formula

$$f(z) - f(u) = \int_{\mathbb{T}} \left\{ \frac{\zeta}{\zeta - z} - \frac{\zeta}{\zeta - u} \right\} f(\zeta) dm(\zeta).$$

Triangle inequality

$$\|f(z) - f(u)\|_X \leq \frac{|z-u|}{(1-|u|)(1-|z|)} \int_{\mathbb{T}} \|f\|_X dm$$

Example: Rudin Shapiro Martingales

Fix a complex sequence (c_n) with $\sum_{k=1}^{\infty} |c_k|^2 \leq 1$.

Define recursively: $F_1 = G_1 = 1$ and for $w = (w_n) \in \mathbb{T}^{\mathbb{N}}$

$$F_{m+1}(w) = F_m(w) + \overline{G_m}(w) c_{m+1} w_{m+1},$$

$$G_{m+1}(w) = G_m(w) - \overline{F_m}(w) c_{m+1} w_{m+1}.$$

Pythagoras for (F_m, G_m) and $(\overline{G}_m, -\overline{F}_m)$ gives

$$|F_{m+1}(w)|^2 + |G_{m+1}(w)|^2 = (1 + |c_{m+1}|^2)(|F_m(w)|^2 + |G_m(w)|^2).$$

and repeat

$$|F_{m+1}(w)|^2 + |G_{m+1}(w)|^2 = \prod_{k=1}^{m+1} (1 + |c_k|^2) 2.$$

Rudin Shapiro Martingales II

$F = (F_n)$ a uniformly bounded Hardy martingale

$$F_n(w) = \sum_{m=1}^{n} \overline{G_m}(w) c_{m+1} w_{m+1}$$

for which the martingale differences reproduce the (c_m).

$$\mathbb{E}_w(\overline{w_m}(F_n(w) - F_{n-1}(w)) = c_{m+1} \mathbb{E}_w \overline{G_m}(w) = c_{m+1}.$$

Rudin Shapiro martingales gives the cotype 2 estimate for L^1/H^1

$$\mathbb{E}_w \| \sum_{m=1}^{n} w_m x_m \|_{L^1/H^1} \geq c \left(\sum \|x_m\|^2_{L^1/H^1} \right)^{1/2}.$$

when the x_m have **well separated** Fourier spectrum.

The Origins I

A. Pelczynski posed **famous problems** in "Banach Spaces of analytic functions and absolutely summing operators, (1977)."

Does H^1 have an unconditional basis?

Does there exist a subspace of L^1/H^1 isomorphic to L^1?

Does L^1/H^1 have cotype 2?

Are the spaces $A(D^n)$ and $A(D^m)$ not isomorphic when $n \neq m$?

The Origins II

Hardy martingales gave rise to the operators by which **Maurey** proved that H^1 has an unconditional basis;

and to the isomorphic invariants by which **Bourgain** proved the dimension conjecture, that L^1/H^1 has cotype 2 and that L^1 embeds into L^1/H^1.

Pisier's L^1/H^1 valued Riesz products form a Hardy martingale that is strongly intertwined with Bourgain's solutions and played an important role for the work of **Garling, Tomczak-Jaegermann, W. Davis** on Hardy martingale cotype and complex uniformly convex renormings of Banach spaces.

Garling's Maximal Functions estimate I .

For any X valued Hardy martingale $F = (F_k)$

$$\mathbb{E}(\sup_{k\in\mathbb{N}} \|F_k\|) \leq e \sup_{k\in\mathbb{N}} \mathbb{E}(\|F_k\|).$$

For any $0 < \alpha \leq 1$, $(\|F_{k-1}\|_X^\alpha)$ is a non- negative sub-martingale

$$\|F_{k-1}\|_X^\alpha \leq \mathbb{E}_{k-1}(\|F_k\|_X^\alpha).$$

Brownian Motion

Let Ω denote the Wiener space $\{z_t : t > 0\}$ denotes complex Brownian Motion started at $0 \in \mathbb{D}$, and define

$$\tau = \inf\{t > 0 : |z_t| > 1\}.$$

For $f \in H^1(\mathbb{T}, X)$, $0 < \alpha < 1$ and $0 < t < \tau$,

$$\|f(z_t)\|_X^\alpha \leq \mathbb{E}(\|f(z_\tau)\|_X^\alpha | \mathcal{F}_t),$$

and

$$\mathbb{E}(\sup_{t<\tau} \|f(z_t)\|_X) \leq e \sup_{t<\tau} \mathbb{E}(\|f(z_t)\|_X),$$

where the integration is over the Wiener space Ω.

Garling's Maximal Functions estimate II .

$$\Sigma = \mathbb{T}^{k-1} \times \Omega, \quad x \in \mathbb{T}^{k-1}, \quad \omega \in \Omega.$$

For any X valued Hardy martingale $F = (F_k)$, the maximal function

$$F_k^*(x, \omega) = \max \left\{ \max_{m \le k-1} \|F_m(x)\|_X, \sup_{t < \tau} \|F_k(x, z_t(\omega))\|_X \right\}$$

satisfies

$$\mathbb{E}_\Sigma(F_k^*) \le e^2 \mathbb{E}(\|F_k\|_X).$$

Davies Decomposition I.

Let $F=(F_k)_{k=1}^n$ be an X valued Hardy martingale.

With the **maximal function estimates,** the standard B. Davies decomposition and **Doob's projection** we obtain a splitting of F into **Hardy martingales**

$$F=G+B$$

satisfying

$$\|\Delta G_k\|_X \leq \max_{m\leq k-1}\|F_m\|_X,$$

and

$$\mathbb{E}(\sum_{k=1}^n\|\Delta B_k\|_X)\leq C\mathbb{E}(\|F\|_X).$$

Sketch of Proof. Fix $x \in \mathbb{T}^{k-1}, v \in \mathbb{T}$. Define

$$f(v) = \Delta F_k(x,v), \qquad \lambda = \max_{m \leq k-1} \|F_m(x)\|_X.$$

and

$$\rho = \inf\{t < \tau : \|f(z_t)\|_X > 2\lambda\}, \quad R_k = f(z_\rho), \quad S_k = f(z_\rho) - f(z_\tau).$$

• $F_k^*(x,\omega) \leq 4(F_k^*(x,\omega) - F_{k-1}^*(x,\omega)), \quad \omega \in A = \{\rho < \tau\}.$

• $\|S_k\|_X \leq 2F_k^* \leq 8(F_k^* - F_{k-1}^*), \quad \sum_{k=1}^n \|S_k\|_X \leq 8F_n^*.$

• By choice of the stopping time ρ, $\|R_k\| \leq 2\lambda$.

Doob's projection generates the analytic functions

$$\Delta B_k = \mathbb{E}(S_k | z_\tau = z), \quad \Delta G_k = \mathbb{E}(R_k | z_\tau = z), \qquad z \in \mathbb{T}.$$

Improved Davies Decomposition (PFXM) A Hardy martingale $F = (F_k)$ can be decomposed into Hardy martingales as $F = G + B$ such that

$$\|\Delta G_k\|_X \le C\|F_{k-1}\|_X,$$

and

$$\mathbb{E}(\sum_{k=1}^{\infty} \|\Delta B_k\|_X) \le C\mathbb{E}(\|F\|_X).$$

Lemma

If $h \in H_0^1(\mathbb{T}, X)$, $z \in X$ there exists $g \in H_0^\infty(\mathbb{T}, X)$ with

$$\|g(\zeta)\|_X \le C_0\|z\|_X, \qquad \zeta \in \mathbb{T}$$

and

$$\|z\|_X + \frac{1}{8}\int_{\mathbb{T}} \|h - g\|_X dm \le \int_{\mathbb{T}} \|z + h\|_X dm.$$

Sketch of Proof. Fix $x \in \mathbb{T}^{k-1}$. Put

$$h(y) = \Delta F_k(x, y) \quad \text{and} \quad z = F_{k-1}(x).$$

Lemma yields a bounded analytic g with

$$\|z\|_X + 1/8 \int_{\mathbb{T}} \|h - g\|_X dm \leq \int_{\mathbb{T}} \|z + h\|_X dm; \quad \|g(\zeta)\|_X \leq C_0 \|z\|_X.$$

Define

$$\Delta G_k(x, y) = g(y), \quad \Delta B_k(x, y) = h(y) - g(y).$$

Then

$$\|F_{k-1}\|_X + 1/8\mathbb{E}_{k-1}(\|\Delta B_k\|_X) \leq \mathbb{E}_{k-1}(\|F_k\|_X).$$

Integrate and take the sum,

$$\sum \mathbb{E}(\|\Delta B_k\|_X) \leq 4 \sup \mathbb{E}(\|F_k\|_X).$$

Iterating Maurey's embedding: An Alternative to Decomposing.

Given $\eta > 0$ and a X valued Hardy martingale (g_k) there exists a vector valued Hardy martingale (G_k), and an increasing sequence of integers

$$m(0) < m(1) < \cdots < m(n) < \ldots \text{ so that:}$$

(G_k), has **small previsible increments**,

$$\|\Delta G_k\|_X \leq \eta\beta_{k-1}, \quad \mathbb{E}(\sup_{k\in\mathbb{N}} \beta_k) \leq \sup_{k\in\mathbb{N}} \mathbb{E}(\|g_k\|_X),$$

and **on the subsequence** $m(n)$ it has almost identitcal L^p norms

$$\mathbb{E}\|g_k\| \overset{1\pm\eta}{\sim} \mathbb{E}\|G_{m(k)}\|, \quad \mathbb{E}\|\Delta g_k\|^p \overset{1\pm\eta}{\sim} \mathbb{E}\|G_{m(k)} - G_{m(k-1)}\|^p.$$

We Continue with scalar valued Martingales
Martingale Norms and Spaces

Let $G = (G_k)$ be an integrable $(\mathcal{F}_n)$ martingale.

$$\|G\|_{\mathcal{P}} = \mathbb{E}\Big(\sum_{k=1}^{n} \mathbb{E}_{k-1} |\Delta G_k|^2 \Big)^{1/2},$$

$$\|G\|_{H^1} = \mathbb{E}\Big(\sum_{k=1}^{n} |\Delta G_k|^2 \Big)^{1/2} \quad \text{and} \quad \|G\|_{\mathcal{A}} = \mathbb{E}\Big(\sum_{k=1}^{n} |\Delta G_k| \Big).$$

The resulting spaces are related as follows

$$\mathcal{A} \subseteq H^1. \qquad \mathcal{P} \subseteq H^1, \qquad H^1 \subseteq \mathcal{P} + \mathcal{A}.$$

Triangle Inequality, Burkholder-Gundy, Davis-Garsia.

Martingale Transforms

Let $(\Omega, (\mathcal{F}_n), \mathbb{P})$ be a filtered probability space.

Let w_k is complex valued, adapted, and $|w_k| \leq 1$. The martingale transforms

$$T(G) = \Im \left[\sum_{k=1}^{n} w_{k-1} \cdot \Delta G_k \right],$$

is a contraction on H^1, as well as on $\mathcal{P}$.

In **general** T **is unbounded** on L^1.

The Transform Estimate: (PFXM)

Let $F = (F_k)$ be a martingale. Define the transform

$$T(G) = \Im\left[\sum w_{k-1} \cdot \Delta G_k\right], \qquad w_{k-1} = \overline{F_{k-1}}/|F_{k-1}|.$$

If G **satisfies** $|\Delta G_k| \le A|F_{k-1}|$, **then**

$$\|T(G)\|_{\mathcal{P}} \le C\|F\|_{L^1}^{1/2}\|F\|_{H^1}^{1/2} + C\|F - G\|_{\mathcal{A}},$$

where $C = C(A)$.

Proof exploits non-linear telescoping.

Davis-Garsia inequalities for Hardy Martingales

PFXM. Every *scalar valued Hardy* martingale $F = (F_k)$ has a decomposition into *Hardy* martingales as $F = G + B$ so that

$$\|G\|_{\mathcal{P}} + \|B\|_{\mathcal{A}} \leq C\|F\|_{L^1}, \qquad |\Delta G_k| \leq C|F_{k-1}|.$$

Compare with the classical Davis-Garsia inequality.

A general martingale $F = (F_k)$ has a decomposition $F = G + B$ so that

$$\|G\|_{\mathcal{P}} + \|B\|_{\mathcal{A}} \leq C\|F\|_{H^1}, \qquad |\Delta G_k| \leq C \max_{m \leq k-1} |F_m|.$$

Proving DGI for Hardy Martingales.

Step 1. Split the Hardy martingale $F = (F_k)$ as

$$F = G + B, \qquad |\Delta G_k| \le A|F_{k-1}|, \qquad \|B\|_{\mathcal{A}} \le C\|F\|_{L^1}.$$

Define the transform

$$T(H) = \Im\left[\sum w_{k-1} \cdot \Delta H_k\right], \qquad w_{k-1} = \overline{F_{k-1}}/|F_{k-1}|.$$

The martingale transform estimate gives

$$\|T(G)\|_{\mathcal{P}} \le C\|F\|_{L^1}^{1/2}\|F\|_{H^1}^{1/2} + C\|B\|_{\mathcal{A}}.$$

Step 2

Since G is a Hardy martingale and $|w_{k-1}| = 1$ we have

$$\mathbb{E}_{k-1}|\Delta G_k|^2 = 2\mathbb{E}_{k-1}|\Im\left[w_{k-1} \cdot \Delta G_k\right]|^2,$$

hence

$$\|G\|_{\mathcal{P}} = \sqrt{2}\|T(G)\|_{\mathcal{P}}, \qquad \text{and} \qquad \|G\|_{\mathcal{P}} \leq C\|F\|_{L^1}^{1/2}\|F\|_{H^1}^{1/2}.$$

A Hardy martingale F has a decomposition into Hardy martingales $F = G + B$ so that

$$\|B\|_{\mathcal{A}} \leq C\|F\|_{L^1}, \qquad \|G\|_{\mathcal{P}} \leq C\|F\|_{L^1}^{1/2}\|F\|_{H^1}^{1/2}.$$

Step 3

By the triangle inequality and Burkholder Gundy inequality the splitting $F = G + B$ with

$$\|G\|_{\mathcal{P}} \le C\|F\|_{L^1}^{1/2}\|F\|_{H^1}^{1/2}, \qquad \|B\|_{\mathcal{A}} \le C\|F\|_{L^1},$$

yields

$$\|F\|_{H^1} \le \|G\|_{H^1} + \|B\|_{H^1} \le \|G\|_{\mathcal{P}} + \|B\|_{\mathcal{A}} \le C\|F\|_{L^1}^{1/2}\|F\|_{H^1}^{1/2}.$$

Cancelling $\|F\|_{H^1}^{1/2}$ gives the square function estimate,

$$\|F\|_{H^1} \le C\|F\|_{L^1},$$

and simultaneously the Davis and Garsia inequality

$$\|G\|_{\mathcal{P}} + \|B\|_{\mathcal{A}} \le C\|F\|_{L^1}.$$

This proof is stable under dyadic perturbations.

THANK YOU!