Principle of local reflexivity, respecting subspaces, and approximation properties of pairs

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X – Banach space, $\lambda \geq 1$

Recall: X has λ-BAP: $\forall E \subset X$, $\dim E < \infty$, $\forall \varepsilon > 0$
$\exists S \in \mathcal{F}(X) := \mathcal{F}(X, X)$ such that $\|S\| \leq \lambda + \varepsilon$ and $Sx = x$, $x \in E$.

Equivalent to ([JRZ]=Johnson, Rosenthal, Zippin; Israel J. Math, 1971): $\exists (S_\alpha) \subset \mathcal{F}(X)$ such that $\limsup \|S_\alpha\| \leq \lambda$ and $S_\alpha \to I_X$ pointwise.

$Y \subset X$, Y – closed subspace

[FJP]=Figiel, Johnson, Pełczyński, Some approximation properties of Banach spaces and Banach lattices, Israel J. Math, 2011:
The pair (X, Y) has λ-BAP: $\forall E \subset X$, $\dim E < \infty$, $\forall \varepsilon > 0$
$\exists S \in \mathcal{F}(X)$ such that $S(Y) \subset Y$, $\|S\| \leq \lambda + \varepsilon$, and $Sx = x$, $x \in E$.

X – λ-BAP $\Leftrightarrow$ (X, X) – λ-BAP $\Leftrightarrow$ $(X, \{0\})$ – λ-BAP

$Y \subset X \Rightarrow Y^{\perp} := \{x^* \in X^* : \; x^*(y) = 0 \;\; \forall y \in Y\} \subset X^*$
$S(Y) \subset Y \Leftrightarrow S^*(Y^{\perp}) \subset Y^{\perp}$

For reflexive X: (X, Y) – BAP$\Leftrightarrow (X^*, Y^{\perp})$ – BAP

In general: (X, Y) – BAP$\not\Rightarrow (X^*, Y^{\perp})$ – BAP
Enflo, James, Lindenstrauss: X – BAP$\not\Rightarrow X^*$ – BAP

Grothendieck (essentially): X – λ-BAP$\Leftarrow X^*$ – λ-BAP
easy with the PLR from [Johnson;TAMS,1971] or [JRZ])

[Oja, Treialt; Stud.Math,2013]: (X, Y) – λ-BAP$\Leftarrow (X^*, Y^{\perp})$ – λ-BAP
the PLR seems not working

[Oja, Treialt, Stud. Math, 2013] :

• (X, Y) – λ-BAP $\Leftarrow$ $(X^*, Y^\perp)$ – λ-BAP
  the PLR seems not working

Proof relies on [Oja, JMAA, 2006]; it does not work if the BAP is given by projections.

• Find a "working" PLR (respecting subspaces) which would give an easy proof and would apply to the case of projections!

BAP given by projections $=:$ π-property

X has π_λ-property: $\forall E \subset X$, $\dim E < \infty$, $\forall \varepsilon > 0$ $\exists$ projection $P \in \mathcal{F}(X)$ such that $\|P\| \leq \lambda + \varepsilon$ and $Px = x$, $x \in E$.

50th anniversary: Lindenstrauss's Memoir, 1964. Important contributions: Michael & Pełczyński, Israel J. Math, 1966 (π_1-property (π_1^∞-property: $\operatorname{ran} P \cong \ell_\infty^n$)); Johnson, Illinois J. Math, 1970; [JRZ]; Pełczyński & Rosenthal, Studia Math, 1975; etc.

Th.1 [JRZ]: (a) $X - \pi$-property $\Leftarrow X^* - \pi$-property

(b) $\left.\begin{array}{l} X - \pi\text{-property} \\ X^* - \text{BAP} \end{array}\right\} \Rightarrow X^* - \pi$-property

Proof relies on a strong form of PLR involving projections.

- Extend JRZ Theorem 1 to (X, Y)!
- Find a PLR respecting subspaces and involving projections!

(X, Y) has π_λ-property: $\forall E \subset X$, $\dim E < \infty$, and $\forall \varepsilon > 0$ $\exists$ projection $P \in \mathcal{F}(X)$ such that $P(Y) \subset Y$, $\|P\| \leq \lambda + \varepsilon$, and $Px = x$, $x \in E$.

(X, Y) has π_λ-duality property: $\forall E \subset X$, $\dim E < \infty$, and $\forall F \subset X^*$, $\dim F < \infty$, and $\forall \varepsilon > 0$ $\exists$ projection $P \in \mathcal{F}(X)$ such that $P(Y) \subset Y$, $\|P\| \leq \lambda + \varepsilon$, and $Px = x$, $x \in E$, and $P^* x^* = x^*$, $x^* \in F$.

$X - \pi_\lambda$-(dual)prop$\Leftrightarrow(X, X) - \pi_\lambda$-(dual)prop$\Leftrightarrow(X, \{0\}) - \pi_\lambda$-(dual)prop

PLR-1 respecting subspaces: X, Z – Banach spaces; $U \subset X$, $V \subset Z$ – closed subspaces; let $S \in \mathcal{F}(Z^*, X^*)$ satisfy $S(V^\perp) \subset U^\perp$.
If $F \subset Z^*$, dim $F < \infty$, and $\varepsilon > 0$, then $\exists T \in \mathcal{F}(X, Z)$ satisfying $T(U) \subset V$ such that

$$\begin{array}{ll} 1^\circ & \big| \|T\| - \|S\| \big| < \varepsilon, \\ 2^\circ & T^* z^* = S z^*,\ z^* \in F, \\ 2^{\circ\circ} & \operatorname{ran} T^* = \operatorname{ran} S, \\ 3^\circ & T^{**} x^{**} = S^* x^{**} \text{ whenever } S^* x^{**} \in Z. \end{array}$$

When $X = Z$ and S is a projection, also T is a projection.

Proof relies on Grothendieck's description $(X \otimes Z)^* = \mathcal{I}(X, Z^*)$, integral operators, equipped with their integral norms.
$(X \otimes Z \subset (\mathcal{F}(X^*, Z), \|\cdot\|)$; $(x \otimes z)(x^*) = x^*(x)z$, $x^* \in X^*$.)
(Via duality $\langle A, x \otimes z \rangle = (Ax)(z)$, $A \in \mathcal{I}(X, Z^*)$.)

Immediate applications of PLR-1 respecting subspaces:

Cor. 1: λ-BAP (π_λ-property) of $(X^*, Y^\perp)$ is given by conjugate operators (projections).

Standard arguments (incl. passing to convex combinations of approximating operators) give a refinement for BAP (but not for π-property):

Cor. 2 [Oja–Treialt]: $(X^*, Y^\perp)$ – λ-BAP$\Rightarrow$ (X, Y) – λ-BdualityAP:
$\forall E \subset X$, $\dim E < \infty$, and $\forall F \subset X^*$, $\dim F < \infty$, and $\forall \varepsilon > 0$
$\exists S \in \mathcal{F}(X)$ such that $S(Y) \subset Y$, $\|S\| \leq \lambda + \varepsilon$, and
$Sx = x$, $x \in E$, and $S^*x^* = x^*$, $x^* \in F$.

>From Cor. 1 and Cor. 2, we get the following Theorem:

Th.: (a) $\left.\begin{array}{l}(X^*, Y^\perp) - \pi_\lambda\text{-prop.} \\ (X, Y) - \mu\text{-BAP}\end{array}\right\} \Rightarrow (X, Y) - \pi_{\lambda\mu+\lambda+\mu}\text{-dual. prop.}$

(b) $\left.\begin{array}{l}(X, Y) - \pi_\lambda\text{-prop.} \\ (X^*, Y^\perp) - \mu\text{-BAP}\end{array}\right\} \Rightarrow (X, Y) - \pi_{\lambda\mu+\lambda+\mu}\text{-dual. prop.}$

Proof: (b) Let $E \subset X$, $\dim E < \infty$, $F \subset X^*$, $\dim F < \infty$, and let $\varepsilon > 0$. Choose $\delta > 0$ such that $(2 + \lambda + \mu)\delta + \delta^2 < \varepsilon$. Cor. 2: (X, Y) – μ-BdualityAP: $\exists S \in \mathcal{F}(X)$ such that $S(Y) \subset Y$, $\|S\| \leq \mu + \delta$, $Sx = x$, $x \in E$, and $S^*x^* = x^*$, $x^* \in F$. $(X, Y) - \pi_\lambda$-prop.: $\exists P \in \mathcal{F}(X)$ such that $P(Y) \subset Y$, $\|P\| \leq \lambda + \delta$, and $Px = x$, $x \in \operatorname{ran} S$. Then $PS = S$. Hence, $Q := P + S - SP \in \mathcal{F}(X)$ is a needed projection.

(a) Similar; uses Cor. 1: π_λ-prop. given by conj. op.

Th.:

(a) $\left.\begin{array}{l}(X^*, Y^\perp) - \pi_\lambda\text{-prop.}\\ (X, Y) - \mu\text{-BAP}\end{array}\right\} \Rightarrow (X, Y) - \pi_{\lambda\mu+\lambda+\mu}\text{-dual. prop.}$

(b) $\left.\begin{array}{l}(X, Y) - \pi_\lambda\text{-prop.}\\ (X^*, Y^\perp) - \mu\text{-BAP}\end{array}\right\} \Rightarrow (X, Y) - \pi_{\lambda\mu+\lambda+\mu}\text{-dual. prop.}$

[Lissitsin, Oja; JMAA, 2011]: If X^* or X^{**} has RNP,
then $(X^*, Y^\perp)$ – AP$\Rightarrow$ $(X^*, Y^\perp)$ – 1-BAP.

Cor.: If X^* or X^{**} has RNP,
then $(X^*, Y^\perp) - \pi_\lambda$-prop. $\Rightarrow$ $(X, Y) - \pi_{2\lambda+1}$-duality prop.

Th. (a), applied 2$\times$: $(X^{**}, Y^{\perp\perp})$ – π_λ-prop. $\Rightarrow$ (X, Y) – π_μ-prop. with $\mu = (\lambda^2 + 2\lambda)^2 + 2(\lambda^2 + 2\lambda)$

Th. 2 [JRZ]: X^{**} – π_λ-prop. $\Rightarrow$ X – π_λ-prop.

We can extend this using
PLR-2 respecting subspaces (weak version): X, Z – Banach spaces; $U \subset X$, $V \subset Z$ – closed subspaces. Let $S \in \mathcal{F}(X^{**}, Z^{**})$ satisfy $S(U^{\perp\perp}) \subset V^{\perp\perp}$. If $E \subset X^{**}$, $\dim E < \infty$, and $F \subset Z^*$, $\dim F < \infty$, and $\varepsilon > 0$, then $\exists T \in \mathcal{F}(X, Z)$ satisfying $T(U) \subset V$ such that

1° $\big| \|T\| - \|S\| \big| < \varepsilon$,

2° $x^{**}(T^* z^*) = (Sx^{**})(z^*)$, $x^{**} \in E$ and $z^* \in F$,

3° $T^{**}x^{**} = Sx^{**}$ whenever $Sx^{**} \in Z$.

When $X = Z$ and S is a projection, also T is a projection.

Proof is immediate: apply 2$\times$ the PLR-1 resp. subsp.

Cor.: $(X^{**}, Y^{\perp\perp})$ – π_λ-prop $\Rightarrow$ (X, Y) – π_λ-prop.

"Moreover" part to the weak version (by "enlarging" argument):

PLR-2 respecting subspaces: X, Z – Banach spaces; $U \subset X$, $V \subset Z$ – closed subspaces. Let $S \in \mathcal{F}(X^{**}, Z^{**})$ satisfy $S(U^{\perp\perp}) \subset V^{\perp\perp}$. If $E \subset X^{**}$, $\dim E < \infty$, and $F \subset Z^*$, $\dim F < \infty$, and $\varepsilon > 0$, then $\exists T \in \mathcal{F}(X, Z)$ satisfying $T(U) \subset V$ such that

1° $\big| \|T\| - \|S\| \big| < \varepsilon$,

2° $x^{**}(T^* z^*) = (Sx^{**})(z^*)$, $x^{**} \in E$ and $z^* \in F$,

3° $T^{**}x^{**} = Sx^{**}$ whenever $Sx^{**} \in Z$.

When $X = Z$ and S is a projection, also T is a projection. Moreover, if $S|_E$ is one-to-one, then also $T^{**}|_E$ is, and

$1^{\circ\circ}$ $\|(T^{**}|_E)^{-1}\| < \|(S|_E)^{-1}\| + \varepsilon$.

PLR in [JRZ]: $X = E \subset Z^{**}$, $S = Id : E \to Z^{**}$, $U = \{0\}$, $V = \{0\}$.

Bellenot's PLR [J. Funct. Anal, 1984]: PLR + $T(E \cap V^{\perp\perp}) \subset V$.
Proof: X, S as above, $U = E \cap V^{\perp\perp}$ ($\Rightarrow S(U^{\perp\perp}) = U \subset V^{\perp\perp}$).

PLR-1 respecting subspaces easily follows from:

Lemma: $G \subset X^*$, $V \subset Z$; let $T \in X \otimes Z^{**}$ satisfy $T(G) \subset V^{\perp\perp}$.
If $F \subset Z^*$, $\dim F < \infty$, then $\exists (T_\alpha) \subset X \otimes Z$ satisfying $T_\alpha(G) \subset V$, for all α, such that

1° $\|T_\alpha\| \to \|T\|$,

2° $T_\alpha^* z^* \to T^* z^*$, $z^* \in Z^*$; and $T_\alpha^* z^* = T^* z^*$, $z^* \in F$, for all α,

3° $T_\alpha x^* = T x^*$ for all α whenever $Tx^* \in Z$.

Recall: $X \otimes Z \subset \mathcal{F}(X^*, Z)$; $(x \otimes z)(x^*) = x^*(x) z, x^* \in X^*$.

Proof of Lemma: (1)
$\mathcal{R} := \{R \in X \otimes Z : R(G) \subset V\}$, $\mathcal{S} := \{S \in X \otimes Z^{**} : S(G) \subset V^{\perp\perp}\}$

$$\mathcal{R}^\perp \subset (X \otimes Z)^* = \mathcal{I}(X, Z^*) \overset{J}{\to} \mathcal{I}(X, Z^{***}) = (X \otimes Z^{**})^* \supset \mathcal{S}^\perp$$

$J(A) = j_{Z^*} A$, $A \in \mathcal{I}(X, Z^*)$, ($j_{Z^*} : Z^* \to Z^{***}$ is can. emb.)
J – isometry into; $J(\mathcal{R}^\perp) \subset \mathcal{S}^\perp$
$\Phi(A + \mathcal{R}^\perp) = J(A) + \mathcal{S}^\perp$, $A \in \mathcal{I}(X, Z^*)$

$$\mathcal{R}^* = \mathcal{I}(X, Z^*)/\mathcal{R}^\perp \overset{\Phi}{\to} \mathcal{I}(X, Z^{***})/\mathcal{S}^\perp = \mathcal{S}^*$$

$\Phi^*(T) \in \|T\| B_{\mathcal{R}^{**}} \Rightarrow \exists (T_\alpha) \subset \mathcal{R}$, i.e., $T_\alpha(G) \subset V$ for all α, with 1°, $T_\alpha^* z^* \to T^* z^*$, $z^* \in Z^*$, and $T_\alpha x^* \to T x^*$ whenever $Tx^* \in Z$.

(2) Make convergences “constant” (where needed) using a perturbation argument from [Oja, Põldvere; PAMS, 2007], inspired by [JRZ].

For the details, please see:

E. Oja, Principle of local reflexivity respecting subspaces, Adv. Math. **258** (2014) 1–12.