Asymmetry of convex sets and rough approximation by polytopes

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In memory of Olek Pełczyński, a teacher and a friend

http://www.cwru.edu/artsci/math/szarek/

Abstract

The primary result we report on is as follows:

If $K \subset \mathbf{R}^n$ is a convex body and $\delta \in (0, \frac{1}{2})$, then there exists a polytope $Q \supset K$ with $N \leq n \exp(C\delta n)$ faces such that $\delta(Q - a) \subset (K - a)$, where a is the centroid of Q.

This implies bounds for the metric entropy of Banach-Mazur compacta on coarse scales and complements recent results of Barvinok, Litvak/Rudelson/Tomczak-Jaegermann and Pisier. Speaker's original interest in this circle of questions stemmed from an inquiry of Olek Pełczyński around the turn of the millennium.

Asymmetry of convex bodies

$K \subset \mathbb{R}^n$, $\quad x \in K$ an interior point

$\rho(K,x) := \max\left\{ \frac{\mathrm{vol}(K \cap D)}{\mathrm{vol}(K \setminus D)} \ : \ D \text{ a halfspace with } x \in \partial D =: H \right\}$

$\rho(K) := \inf_x \rho(K,x)$

Problem: Find (or upper-bound) $\quad \sup_{K \subset \mathbb{R}^n} \rho(K)$

Answer: $\quad \left(\frac{n+1}{n}\right)^n - 1 < e - 1 < 1.7183$

Moreover, the same holds for $\rho(K,a)$ when a is the centroid of K.

Reference: B. Grünbaum, *Partitions of mass-distributions and of convex bodies by hyperplanes*, Pacific J. Math. 10 (1960), 1257-1261.

Sketch of the proof

First, we have an equality iff K is a "pyramid," i.e.,

$$K = \mathrm{conv}(\{v\} \cup B),$$

where B is an $n-1$-dimensional convex body (the "base") and v the vertex, and H is parallel to B, and passes through the centroid of K.

Next, it is not hard to show that any other configuration can not be extremal and since – by compactness – $\sup_{K \subset \mathbb{R}^n} \rho(K)$ is attained, it must be attained on pyramids.

The extremal bodies: s-concave functions

Assume the centroid of K is at the origin and let $H = \theta^\perp$, where $\theta \in S^{n-1}$. Denote

$$\phi(t) := \mathrm{vol}_{n-1}\big(K \cap (H + t\theta)\big). \qquad (*)$$

By the Brunn-Minkowski theorem, the function $t \to \phi(t)^{1/(n-1)}$ is concave on its support $[-b, c]$. The centroid assumption implies $\int t\phi(t)\,dt = 0$.

Conversely, for any $\phi \geq 0$, $\phi \not\equiv 0$ with these properties one may construct a body K so that $(*)$ holds and that the centroid of K is at the origin.

Finally, we notice that unless $\phi(t)^{1/(n-1)}$ is affine with $\phi(c) = 0$, ϕ can be modified so that $\int_{t\geq 0} \phi(t)\,dt / \int \phi(t)\,dt$ strictly decreases.

Comments

Similar arguments were used by Fradelizi and Fradelizi/Guedon (2000, 2004), from which the conclusion can be likely formally derived. Indeed, the observation was presumably made independently many times over the last 100 (or at least 50+) years.

A rephrasing of the result is that the convex floating body $K_t \neq \emptyset$ if $t \leq 1/e$; similar results have been part of folklore for some time.

It turned out that the characterization of the equality case (as "pyramids," formally different than in Grünbaum's paper) was relevant to recent work by Nill/Paffenholz (2012) on toric varieties. After the talk, B. Mityagin pointed out that he published the same result – with the same characterization – in 1969, following on the earlier work of Levin (1962, 1965).

Forward to 2008-2013

Barvinok/Veomett (2008)
The computational complexity of convex bodies

Barvinok (2012)
Thrifty approximations of convex bodies by polytopes

Litvak/Rudelson/Tomczak-Jaegermann (2012)
On approximations by projections of polytopes with few facets

Pisier (2013)
On the metric entropy of the Banach-Mazur compactum

nth Banach-Mazur compactum: the space of n-dimensional normed spaces or, equivalently, n-dimensional (symmetric) convex bodies, endowed with the (multiplicative) distance

$$d(K, L) := \inf\{\lambda \ : \ \exists T \in GL(n) \ K \subset T(L) \subset \lambda K\}$$

Why are these related, and why are they interesting?

If $K \subset \mathbb{R}^n$ can be well-approximated by an affine image of a polytope with few facets (say, polynomial in n), then many questions about K can be reduced to doable linear programming problems.

Equivalent to approximability by a projection of a section of a simplex of not-to-large dimension

L/R/T-J (2012): Not always possible, even for symmetric bodies, because that would imply too-good-to-be-true upper bounds for the metric entropy (i.e., covering numbers) of the Banach-Mazur compactum at coarse resolutions.

P (2013): primarily interested in the operator space version, but taking the opportunity to clarify the "classical" picture that was further confused by misleading statements in the literature; bounds not sharp at coarse resolutions

Constructive Approximation

Ben-Tal/Nemirovski (2001)
On polyhedral approximations of the second-order cone

A reasonably explicit approximation of the n-dimensional Euclidean ball by a projection of a section of a simplex of dimension $O(n)$

A similar feat with projection of a cube, or section of the ℓ_1-ball, would be a major achievement and would have important algorithmic implications, including for compressed sensing.

Incidentally, in these cases the one step procedure is (essentially) equivalent to the two step procedure projection of a section or, equivalently, section of a projection.

Rough approximation, coarse embeddings, and entropy at coarse resolutions

B (2012): Given $n \in \mathbb{N}$ and $r \in [2, n^{1/2}]$, every n-dimensional normed space r-embeds in ℓ_∞^N with $N \leq \exp(Cn \log r / r^2)$. This implies upper bounds for covering numbers of the Banach-Mazur compactum at coarse resolutions.

L/R/T-J (2012): Lower bounds for covering numbers of the Banach-Mazur compactum at coarse resolutions.

New in this report: Upper bounds in the non-symmetric case and much simpler arguments than in B (2012), at the price of worse dependence on r. Also, tying up loose ends from L/R/T-J (2012).

Both B (2012) and the present approach also work on the non-coarse (near- and almost-isometric) scale, but not emphasized here.

Result from the abstract, corollary

Theorem *If* $K \subset \mathbf{R}^n$ *is a convex body and* $\delta \in (0, \frac{1}{2})$*, then there exists a polytope* $Q \supset K$ *with* $N \leq n \exp(C\delta n)$ *faces such that* $\delta(Q - a) \subset (K - a)$*, where* a *is the centroid of* Q*. If* K *is symmetric, so can be chosen* Q*. Similar results hold for polytopes with* N *vertices.*

Corollary *Given* $r > 2$ *and* $n \in \mathbb{N}$*, the space of* n*-dimensional convex bodies endowed with the Banach-Mazur "distance" admits an* r*-net of cardinality* $\leq \exp\big(\exp(Cn/r)\big)$*.*

Proof of the Corollary The Theorem reduces the problem to subspaces of ℓ_∞^N. It remains to use known estimates for metric entropy of the Grassmann manifold $G_{N,n}$.

Sketch of the proof, the "few vertices" variant

First, let us pretend K is symmetric and $Y = (\mathbb{R}^n, \|\cdot\|_K)$. We choose vertices using a greedy algorithm.

Step 1, Initiatization: Choose $y_1, y_2, \ldots, y_n$ to be the Auerbach basis in Y and $y_{n+k} = -y_k, k = 1, 2, \ldots, n$.

Step 2, "Greedy Induction": Once $y_1, y_2, \ldots, y_m$, $m \geq 2n$ are selected, set $P_m := \mathrm{conv}\{\pm y_j : 1 \leq j \leq m\}$ and chose as y_{m+1} the point $y \in K$ that maximizes $\|y\|_{P_m}$.

Claim *If $\|y_{m+1}\|_{P_m} \geq \delta^{-1}$, then*

$$\mathrm{vol}(P_m) \leq \big(1 - (1-\delta)^n\big)\mathrm{vol}(P_{m+1}).$$

Hence $\|y_{m+1}\|_{P_m} \geq \delta^{-1}$ can not happen too many times.

Proof of the Claim is based on

Lemma *Let* $\psi : [0,1] \to [0,\infty)$ *be concave and let* $p > 0$. *Then, for every* $\delta \in (0,1)$,

$$\int_{\delta}^{1} \psi(t)^p \, dt \geq (1-\delta)^{p+1} \int_{0}^{1} \psi(t)^p \, dt$$

The non-symmetric case requires appealing to $\rho(K) < e - 1$.

A loose end from L/R/T-J (2012)

Problem *Is there L with (say)* $\dim L = poly(n)$ *such that every (say, symmetric) n-dimensional body K can be well-approximated by an affine image of a section of L?*

Answer

NO if sections of L are required to be "reasonably central."

YES if there are no restrictions on sections.

THANK YOU