Entropy numbers and eigenvalues of operators

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Spectral radius formula

- The spectrum $\sigma(T)$ of T on a complex Banach space X

$$\sigma(T) := \left\{ \lambda \in \mathbb{C} \ ; \ \lambda I_X - T \ \text{ is not invertible in } \ L(X) \right\}$$

- The essential spectrum

$$\sigma_{ess}(T) := \sigma(\overline{T})$$

$\overline{T}$ is the coset of T in the Calkin algebra $L(X)/K(X)$.

- Gelfand's spectral radius formula

$$r(T) := \sup_{\lambda \in \sigma(T)} |\lambda| = \lim_{m \to \infty} \| T^m \|^{1/m}$$

- The essential spectral radius

$$r_{ess}(T) := r(\overline{T}) = \lim_{m \to \infty} \| T^m \|_{ess}^{1/m}$$

Eigenvalue sequence

The Riesz part of the spectrum $\Lambda(T)$ is at most countable and consists of isolated eigenvalues of finite algebraic multiplicity.

$$\Lambda(T) := \left\{ \lambda \in \sigma(T) \ ; \ |\lambda| > r_{ess}(T) \right\}$$

We assign an eigenvalue sequence $\left\{\lambda_n(T)\right\}_{n=1}^{\infty}$ for $T \in L(X)$ from the elements of the set $\Lambda(T) \cup \left\{r_{ess}(T)\right\}$ as follows:

- The eigenvalues are arranged in an order of non-increasing absolute values.
- Every eigenvalue $\lambda \in \Lambda(T)$ is counted according to its algebraic multiplicity.
- If T possesses less than n eigenvalues λ with $|\lambda| > r_{ess}(T)$, we let
$$\lambda_n(T) = \lambda_{n+1}(T) = \ldots = r_{ess}(T)$$

Entropy numbers

Definition

The n-th entropy number $\varepsilon_n(T)$ of $T \in L(X, Y)$ is defined by

$$\varepsilon_n(T) := \inf\left\{ \varepsilon > 0 \;\; ; \;\; T(U_X) \subset \bigcup_{i=1}^{n} \{y_i + \varepsilon U_Y\}, \quad y_i \in Y \right\}$$

- Entropy numbers are monotonne

$$0 \leqslant \ldots \leqslant \varepsilon_3(T) \leqslant \varepsilon_2(T) \leqslant \varepsilon_1(T) = \|T\|$$

- The measure of non-compactness

$$\beta(T) := \lim_{n \to \infty} \varepsilon_n(T)$$

Carl-Triebel's inequality (1980)

Let $\{\lambda_n(T)\}_{n=1}^{\infty}$ be an eigenvalue sequence of $T \in L(X)$ on a complex Banach space X.

- Carl's inequality

$$|\lambda_n(T)| \leqslant \sqrt{2}\, e_n(T) \quad \text{where} \quad e_n(T) := \varepsilon_{2^{n-1}}(T)$$

- Carl-Triebel's inequality

$$\left(\prod_{i=1}^{n} |\lambda_i(T)|\right)^{1/n} \leqslant \inf_{k \in \mathbb{N}} k^{1/(2n)} \varepsilon_k(T)$$

Banach couple

- We call $\vec{A} := (A_0, A_1)$ a Banach couple if both A_0 and A_1 are Banach spaces such that

$$A_0, A_1 \hookrightarrow \mathcal{X}$$

For a given Banach couple $\vec{A}$, we define spaces

- intersection $A_0 \cap A_1$ with the norm

$$\|a\|_{A_0 \cap A_1} = \max\left\{\|a\|_{A_0}, \|a\|_{A_1}\right\}$$

- sum $A_0 + A_1$ with the norm

$$\|a\|_{A_0 + A_1} = \inf_{a = a_0 + a_1}\left\{\|a_0\|_{A_0} + \|a_1\|_{A_1}\right\}$$

Interpolation functor

- By $T\colon \vec{A} \to \vec{B}$ we denote an operator $T\colon A_0 + A_1 \to B_0 + B_1$, such that
$$T|_{A_j} \in L(A_j, B_j), \quad j = 0, 1$$

Definition

By an interpolation functor we mean a mapping $\mathcal{F}\colon \vec{\mathcal{B}} \to \mathcal{B}$

- $A_0 \cap A_1 \subset \mathcal{F}(\vec{A}) \subset A_0 + A_1$ for any $\vec{A} \in \vec{\mathcal{B}}$
- $T\big(\mathcal{F}(\vec{A})\big) \subset \mathcal{F}(\vec{B})$ for any $\vec{A}, \vec{B} \in \vec{\mathcal{B}}$ and $T\colon \vec{A} \to \vec{B}$

Interpolation functor of exponential type of θ

For all interpolation functors $\mathcal{F}$

$$\|T\|_{\mathcal{F}(\vec{A})\to\mathcal{F}(\vec{B})} \leqslant C \max\{\|T\|_{A_0\to B_0}, \|T\|_{A_1\to B_1}\}$$

If in addition there exists $\theta \in (0,1)$ such that

$$\|T\|_{\mathcal{F}(\vec{A})\to\mathcal{F}(\vec{B})} \leqslant C \|T\|_{A_0\to B_0}^{1-\theta} \|T\|_{A_1\to B_1}^{\theta},$$

then $\mathcal{F}$ is called of exponential type of θ.

- The real $\mathcal{F}(\cdot) = (\cdot)_{\theta,q}$ and complex $\mathcal{F}(\cdot) = [\cdot]_\theta$ interpolation functors are of exponential type of θ.

Recollect...

- The n-th entropy number $\varepsilon_n(T)$ of $T \in L(X,Y)$

$$\varepsilon_n(T) := \inf\left\{ \varepsilon > 0 \;\; ; \;\; T(U_X) \subset \bigcup_{i=1}^{n} \{y_i + \varepsilon U_Y\}, \quad y_i \in Y \right\}$$

- The measure of non-compactness

$$\beta(T) := \lim_{n\to\infty} \varepsilon_n(T)$$

Interpolation of the measure of non-compactness β

A delicate problem

Let $\mathcal{F}$ be an interpolation functor of exponential type of θ. Does there exist a constant $C > 0$ such that for any $T\colon \vec{A} \to \vec{B}$

$$\beta\big(T\colon \mathcal{F}(\vec{A}) \to \mathcal{F}(\vec{B})\big) \leqslant C\,\beta\big(T\colon A_0 \to B_0\big)^{1-\theta}\beta\big(T\colon A_1 \to B_1\big)^{\theta} \text{ ?}$$

This question was answered positively

- for the real interpolation functor $\mathcal{F}(\cdot) = (\cdot)_{\theta,q}$ by Cobos, Fernández-Martínez and Martínez (1999), R.S. (2006),
- for the complex interpolation functor $\mathcal{F}(\cdot) = [\cdot]_\theta$ in the case where $\vec{B}$ satisfies an approximation condition by Teixeira and Edmunds (1981), R.S. (2014).

Interpolation of entropy numbers fails

A more delicate problem

Let $\mathcal{F}$ be an interpolation functor of exponential type of θ. Does there exist a constant $C > 0$ such that for any $T\colon \vec{A} \to \vec{B}$

$$\varepsilon_{k_0 k_1}\big(T\colon \mathcal{F}(\vec{A}) \to \mathcal{F}(\vec{B})\big) \leqslant C\, \varepsilon_{k_0}(T\colon A_0 \to B_0)^{1-\theta}\, \varepsilon_{k_1}(T\colon A_1 \to B_1)^{\theta} \text{ ?}$$

- This question was answered negatively for the real interpolation functor $\mathcal{F}(\cdot) = (\cdot)_{\theta,q}$ by Edmunds and Netrusov (2011).

The reduction

$$\vec{B} = \vec{A}$$

Recollect...

- The n-th entropy number $\varepsilon_n(T)$ of $T \in L(X)$

$$\varepsilon_n(T) := \inf\left\{ \varepsilon > 0 \;\; ; \;\; T(U_X) \subset \bigcup_{i=1}^{n} \{x_i + \varepsilon U_X\}, \quad x_i \in X \right\}$$

- Carl's inequality

$$|\lambda_n(T)| \leqslant \sqrt{2}\, e_n(T) \quad \text{where} \quad e_n(T) := \varepsilon_{2^{n-1}}(T)$$

- Carl-Triebel's inequality

$$\left(\prod_{i=1}^{n} |\lambda_i(T)| \right)^{1/n} \leqslant \inf_{k \in \mathbb{N}} k^{1/(2n)} \varepsilon_k(T)$$

Interpolation variant of Carl-Triebel's inequality

Theorem (2013)

Suppose that $\mathcal{F}$ is an interpolation functor of exponential type of θ. If $T: \vec{A} \to \vec{A}$, then

$$\left|\lambda_n\Big(T|_{\mathcal{F}(\vec{A})}\Big)\right| \leqslant 2e_n(T|_{A_0})^{1-\theta}\, e_n(T|_{A_1})^{\theta}$$

and

$$\left(\prod_{i=1}^{n}\left|\lambda_i\Big(T|_{\mathcal{F}(\vec{A})}\Big)\right|\right)^{1/n} \leqslant \inf_{k_0,k_1\in\mathbb{N}} (k_0k_1)^{1/2n}\,\varepsilon_{k_0}(T|_{A_0})^{1-\theta}\,\varepsilon_{k_1}(T|_{A_1})^{\theta}$$

Generalizations of the spectral radius formula (1)

- Gelfand's spectral radius formula

$$|\lambda_1(T)| = \lim_{m\to\infty} \|T^m\|^{1/m}$$

Definition

Given $T \in L(X)$, the n-th *approximation number* is defined by

$$a_n(T) := \inf\{ \|T - S\| \ ; \ S \in L(X), \ \mathrm{rank}(S) < n \}$$

- König's (1978) formula; a generalization for higher eigenvalues

$$|\lambda_n(T)| = \lim_{m\to\infty} a_n(T^m)^{1/m}$$

Generalizations of the spectral radius formula (2)

- The n-th entropy modulus $g_n(T)$ of $T \in L(X)$ is given by

$$\left(\prod_{i=1}^{n} |\lambda_i(T)|\right)^{1/n} \leqslant \inf_{k \in \mathbb{N}} k^{1/(2n)} \varepsilon_k(T) =: g_n(T)$$

- Makai-Zemánek's formula (1982)

$$\left(\prod_{i=1}^{n} |\lambda_i(T)|\right)^{1/n} = \lim_{m \to \infty} g_n(T^m)^{1/m}$$

Problem

In what form does it exist a formula for the spectral radius of T using the entropy numbers of powers of operators?

Main results - spectral entropy (1)

Theorem (2013)

Let X be a complex Banach space and $T \in L(X)$. If $\{\lambda_n(T)\}$ is an eigenvalue sequence of T, then

$$\sup_{n\in\mathbb{N}} k^{-1/(2n)} \left(\prod_{i=1}^{n} |\lambda_i(T)| \right)^{1/n} = \lim_{m\to\infty} \varepsilon_{k^m}(T^m)^{1/m}$$

Definition

We define the k-th *spectral entropy number* $\mathcal{E}_k(T)$ by

$$\mathcal{E}_k(T) := \lim_{m\to\infty} \varepsilon_{k^m}(T^m)^{1/m} \leqslant \varepsilon_k(T)$$

Main results - spectral entropy (2)

Theorem (2013)

Fix $t \in [1, \infty)$. *If* $\{t_m\} \subset \mathbb{N}$ *is such that* $\lim_{m\to\infty} t_m^{1/m} = t$, *then*

$$\sup_{n\in\mathbb{N}} t^{-1/(2n)} \left(\prod_{i=1}^{n} |\lambda_i(T)| \right)^{1/n} = \lim_{m\to\infty} \varepsilon_{t_m}(T^m)^{1/m}$$

Definition

Define the *spectral entropy map* $t \mapsto \mathcal{E}_t(T)$ of T as follows

$$\mathcal{E}_t(T) := \lim_{m\to\infty} \varepsilon_{t_m}(T^m)^{1/m}$$

Main results - spectral entropy (3)

Proposition

Let X be a complex Banach space and $T \in L(X)$. If $\{\lambda_n(T)\}$ is an eigenvalue sequence of T, then

$$\lim_{m\to\infty} \varepsilon_m(T^m)^{1/m} = \mathcal{E}_1(T) = |\lambda_1(T)|$$

$$\lim_{m\to\infty} e_m(T^m)^{1/m} = \mathcal{E}_2(T) = \sup_{n\in\mathbb{N}} \left(\frac{\prod_{i=1}^{n} |\lambda_i(T)|}{\sqrt{2}} \right)^{1/n}$$

$$\lim_{t\to\infty} \mathcal{E}_t(T) = r_{ess}(T)$$

Entropy moduli

Definition

Let $\varphi \colon [0,\infty) \to [0,\infty)$ be a sub-multiplicative function.
Given an operator $T \in L(X)$ on a complex Banach space X, we define the entropy modulus $g_{s,\varphi}(T)$ as follows

$$g_{s,\varphi}(T) := \inf_{k \in \mathbb{N}} k^{1/(2s)} \varphi\left(\varepsilon_k(T)\right), \qquad s \in (0,\infty)$$

- Denote by $\widetilde{\varphi}$ the function on $[0,\infty)$ given by

$$\widetilde{\varphi}(u) := \lim_{m \to \infty} \varphi(u^m)^{1/m}, \qquad u \geqslant 0$$

- $\widetilde{\varphi}$ is sub-multiplicative and $\widetilde{\varphi} \leqslant \varphi$

Makai-Zemánek's formula revisited

Theorem (2013)

Let X be an arbitrary complex Banach space and $T \in L(X)$. Assume that $\varphi\colon [0,\infty) \to [0,\infty)$ is a nondecreasing, sub-multiplicative and right-continuous function. Then

$$\inf_{t\in[1,\infty)} t^{1/(2s)}\widetilde{\varphi}(\mathcal{E}_t(T)) = \lim_{m\to\infty} g_{s,\varphi}(T^m)^{1/m}\,, \quad s \in (0,\infty)$$

In particular,

$$\inf_{t\in[1,\infty)} t^{1/(2n)}\mathcal{E}_t(T) = \left(\prod_{i=1}^{n} |\lambda_i(T)|\right)^{1/n}$$

Interpolation of spectral entropy numbers holds

Theorem (2013)

If $\mathcal{F}$ be an interpolation functor of exponential type of θ, then for any $T\colon \vec{A} \to \vec{A}$

$$\mathcal{E}_{k_0 k_1}\big(T\colon \mathcal{F}(\vec{A}) \to \mathcal{F}(\vec{A})\big) \leqslant \mathcal{E}_{k_0}(T\colon A_0 \to A_0)^{1-\theta}\, \mathcal{E}_{k_1}(T\colon A_1 \to A_1)^{\theta}$$