Renorming Banach spaces with greedy basis.

András Zsák

Peterhouse, Cambridge

(Joint work with S. J. Dilworth, D. Kutzarova, E. Odell, Th. Schlumprecht.)

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Approximating signals

We are given a Banach space X with a basis (e_i).

Given $x \in X$, we want to approximate x by a linear combination of the e_i.

We also seek an algorithm that produces for any $m \in \mathbb{N}$, a good m-term approximant.

We measure the efficiency of any such algorithm against the smallest theoretical error:

$$\sigma_m(x) = \inf \left\{ \left\| x - \sum_{i \in A} a_i e_i \right\| : \ A \subset \mathbb{N}, \ |A| \leq m, \ (a_i)_{i \in A} \subset \mathbb{R} \right\} .$$

The greedy algorithm and greedy bases

Let $x \in X$ and write $x = \sum x_i e_i \in X$. We fix a permutation $\rho = \rho_x$ of $\mathbb{N}$ such that $|x_{\rho(1)}| \geq |x_{\rho(2)}| \geq \dots$. We then define the m^{th} *greedy approximant to* x by

$$\mathcal{G}_m(x) = \sum_{i=1}^{m} x_{\rho(i)} e_{\rho(i)} \ .$$

We say (e_i) is a *greedy basis* for X if there exists $C > 0$ (C*-greedy*) such that

$$\|x - \mathcal{G}_m(x)\| \leq C\sigma_m(x) \qquad \text{for all } x \in X \text{ and for all } m \in \mathbb{N} \ .$$

The smallest C is the *greedy constant* of the basis.

We shall often assume that (e_i) is normalized: $\|e_i\| = 1$ for all $i \in \mathbb{N}$.

Greedy characterization

A basis (e_i) is said to be *unconditional* if there is a constant K such that

$$\Big\| \sum a_i e_i \Big\| \leq K \cdot \Big\| \sum b_i e_i \Big\| \qquad \text{whenever } |a_i| \leq |b_i| \text{ for all } i \in \mathbb{N} \ .$$

We also say (e_i) is K*-unconditional*. Can always renorm so that $K = 1$ works.

A basis (e_i) is said to be *democratic* if there is a constant Δ such that

$$\Big\| \sum_{i \in A} e_i \Big\| \leq \Delta \Big\| \sum_{i \in B} e_i \Big\| \qquad \text{whenever } |A| \leq |B| \ .$$

We will use the term Δ*-democratic*.

Theorem [S. V. Konyagin, V. N. Temlyakov, '99]
A basis is greedy if and only if it is unconditional and democratic.

If (e_i) is 1-unconditional, then $\Delta \leq C \leq 1 + \Delta$.

Examples

1. The unit vector basis of ℓ_p $(1 \le p < \infty)$ or c_0 is 1-greedy.

2. Any orthonormal basis of a separable Hilbert space is 1-greedy.

3. The Haar basis of $L_p[0,1]$ $(1 < p < \infty)$ is greedy [V. N. Temlyakov, '98].

4. The Haar system in one-dimensional dyadic Hardy space $H_p(\mathbb{R})$, $0 < p < \infty$. [P. Wojtaszczyk, '00].

5. $\left(\oplus_{n=1}^{\infty} \ell_p^n \right)_{\ell_q}$ has a greedy basis whenever $1 \le p \le \infty$ and $1 < q < \infty$ [S. J. Dilworth, D. Freeman, E. Odell and Th. Schlumprecht, '11].

6. None of the space $\left(\bigoplus_{n=1}^{\infty} \ell_p\right)_{\ell_q}$, $1 \le p \ne q < \infty$, $\left(\bigoplus_{n=1}^{\infty} \ell_p\right)_{\mathrm{c}_0}$, $1 \le p < \infty$, and $\left(\bigoplus_{n=1}^{\infty} \mathrm{c}_0\right)_{\ell_q}$, $1 \le q < \infty$, have greedy bases [G. Schechtman, '14].

Questions

Let X be a Banach space with a greedy basis (e_i).

Q1. Can X be renormed so that (e_i) is C-greedy in the new norm, where C is a universal constant? YES with $C = 2 + \varepsilon$.

Q2. Can we take $C = 1$ in Q1? NO in general. Can take $C = 1 + \varepsilon$ for certain bases.

Note: WLOG (e_i) is normalized and 1-unconditional. Recall: $\Delta \leq C \leq 1 + \Delta$.

Q3. Can X be renormed so that (e_i) is Δ-democratic in the new norm, where Δ is a universal constant? YES with $\Delta = 1 + \varepsilon$.

Q4. Can we take $\Delta = 1$ in Q3? NO in general: *e.g.*, the unit vector basis of Tsirelson space T or the Haar basis of dyadic Hardy space H_1 [Dilworth, Odell, Schlumprecht, Z, 11]. YES for certain bases.

Bidemocratic bases.

Introduced by S. J. Dilworth, N. J. Kalton, D. Kutzarova and V. N. Temlyakov.

The *fundamental function* φ of a basis (e_i) of a Banach space X is defined by

$$\varphi(n) = \sup_{|A| \leq n} \Big\| \sum_{i \in A} e_i \Big\| .$$

E.g., for ℓ_p or L_p ($p < \infty$) we have $\varphi(n) \sim n^{1/p}$.

Note: $\varphi(n)$ is increasing and that $(\varphi(n)/n)$ is decreasing.

The *dual fundamental function* φ^* of (e_i) is the fundamental function of (e_i^*).

Note that $n = \big\langle \sum_{i=1}^n e_i, \sum_{i=1}^n e_i^* \big\rangle \leq \varphi(n)\varphi^*(n)$.

We say that (e_i) is *bidemocratic* if there is a constant $\Delta \geq 1$ (Δ-*bidemocratic*) such that $\varphi(n)\varphi^*(n) \leq \Delta n$ for all $n \in \mathbb{N}$.

If (e_i) is bidemocratic with constant Δ, then both (e_i) and (e_i^*) are democratic with constant Δ [DKKT, '03].

Bidemocratic case

Suppose that (e_i) is a greedy and bidemocratic basis for a Banach space X.

Theorem 1 [DOSZ, '11] There is an equivalent norm on X in which (e_i) is normalized, 1-unconditional and 1-bidemocratic. In particular, (e_i) and (e_i^*) are 1-democratic and 2-greedy.

Remark: The implication "1-democratic $\Rightarrow$ 2-greedy" is sharp.

Theorem 2 [DKOSZ] For all $\varepsilon > 0$ there is an equivalent norm on X in which (e_i) is normalized, 1-unconditional, 1-bidemocratic and $(1+\varepsilon)$-greedy.

Notation: a) $\langle x, x^* \rangle = x^*(x)$ for $x \in X$, $x^* \in X^*$

b) $\mathbf{1}_A$ will denote either $\sum_{i \in A} e_i$ or $\sum_{i \in A} e_i^*$. *E.g.*, $\|\mathbf{1}_A\| = \left\| \sum_{i \in A} e_i \right\|$ and $\|\mathbf{1}_A\|^* = \left\| \sum_{i \in A} e_i^* \right\|$.

c) Let $x = \sum x_i e_i \in X$. We set $|x| = \sum |x_i| e_i$, and write $x \geq 0$ if $x_i \geq 0$ for all i.

Proof of Theorem 1

WLOG (e_i) is normalized and 1-unconditional. Let Δ be the bidemocracy constant.

Define $|||x||| = \|x\| \vee \sup \left\{ \left\langle |x|, \frac{\varphi(n)}{n} \mathbf{1}_A \right\rangle : n \in \mathbb{N},\ A \subset \mathbb{N},\ n = |A| \right\}$.

$\left\| \frac{\varphi(n)}{n} \mathbf{1}_A \right\|^* \le \frac{\varphi(n)}{n} \varphi^*(n) \le \Delta$. So $\|x\| \le |||x||| \le \Delta \|x\|$.

Let $|E| = n$. Then $|||\mathbf{1}_E||| \ge \left\langle \mathbf{1}_E, \frac{\varphi(n)}{n} \mathbf{1}_E \right\rangle = \varphi(n)$. For $|A| = m$, we have

$$\left\langle \mathbf{1}_E, \frac{\varphi(m)}{m} \mathbf{1}_A \right\rangle = \frac{\varphi(m)}{m} |E \cap A| \le \frac{\varphi(|E \cap A|)}{|E \cap A|} |E \cap A| \le \varphi(n) \ .$$

So $|||\mathbf{1}_E||| = \varphi(n)$ and $\left|\left|\left| \frac{\varphi(n)}{n} \mathbf{1}_E \right|\right|\right|^* = 1$. QED

Remark: Assume, instead of bidemocracy, that for all $q \in (0,1)$ there exists $C > 0$ such that

$$\mathcal{A} = \left\{ A \subset \mathbb{N} :\ n = |A| < \infty,\ \left\| \frac{\varphi(n)}{n} \mathbf{1}_A \right\|^* \le C \right\}$$

satisfies: $\forall$ finite $E \subset \mathbb{N}$ there exists $A \in \mathcal{A}$ such that $A \subset E$ and $|A| \ge q|E|$.

Characterizing C-greedy bases

Given vectors $x = \sum x_i e_i$ and $y = \sum y_i e_i$ in X, we say y *is a greedy rearrangement of* x if it is obtained from x by rearranging and possibly changing the sign of some of the coefficients of x of maximum modulus.

$x = (-2, \quad 0, 3, 3, 1, 0, 0, -1, 2, -3, 0, 0, \dots)$ and
$y = (-2, -3, 0, 3, 1, 0, 3, -1, 2, \quad 0, 0, 0, \dots)$.

We say (e_i) *has Property (A) with constant* C if for all x, y we have $\|y\| \leq C\|x\|$ whenever y is a greedy rearrangement of x.

Theorem [Albiac, Wojtaszczyk, '06] If (e_i) is 1-unconditional, then (e_i) is C-greedy if and only if it satisfies Property (A) with constant C.

Proof of AW-characterization

Assume (e_i) is 1-unconditional and has Property (A) with constant C.

Fix $x = \sum x_i e_i$ and $m \in \mathbb{N}$. Write $\mathcal{G}_m(x) = \sum_{i\in A} x_i e_i$. Let $s = \min\{|x_i| : i \in A\}$, and note that $|x_i| \leq s \leq |x_j|$ for $i \notin A$, $j \in A$.

Let $b = \sum_{i \in B} b_i e_i$ be an arbitrary m-term approximation.

$$\|x - b\| = \Big\| \sum_{i\in A\setminus B} x_i e_i + \sum_{i \in B} (x_i - b_i) e_i + \sum_{i \notin A\cup B} x_i e_i \Big\|$$

$$\geq \Big\| \sum_{i\in A\setminus B} s e_i + \sum_{i\notin A\cup B} x_i e_i \Big\| \geq \frac{1}{C} \Big\| \sum_{i\in B\setminus A} s e_i + \sum_{i\notin A\cup B} x_i e_i \Big\|$$

$$\geq \frac{1}{C} \Big\| \sum_{i\in B\setminus A} x_i e_i + \sum_{i\notin A\cup B} x_i e_i \Big\| = \frac{1}{C} \|x - \mathcal{G}_m(x)\| \,. \quad \text{QED.}$$

Proof of Theorem 2

Let (e_i) be a greedy and bidemocratic basis of a Banach space X.

Theorem 2 [DKOSZ] For all $\varepsilon > 0$ there is an equivalent norm on X in which (e_i) is normalized, 1-unconditional, 1-bidemocratic and $(1+\varepsilon)$-greedy.

WLOG (e_i) is normalized, 1-unconditional and 1-bidemocratic (by Theorem 1). Define

$$|||x||| = \sup\left\{\left\langle |x|, x^* + \tfrac{1}{\varphi^*(n)}\mathbf{1}_A\right\rangle :\ x^* \in \varepsilon B_{X^*},\ n \in \mathbb{N},\ A \subset \mathbb{N},\ |A| = n\right\}.$$

Calculation shows that (e_i) has Property (A) with constant $1+\varepsilon$, and hence it is $(1+\varepsilon)$-greedy. A little more work . . . QED.

The Upper Regularity Property (URP)

We say that (e_i) has the URP if there exists $0 < \beta < 1$ and $C > 0$ such that

$$\varphi(n) \leq C\left(\tfrac{n}{m}\right)^{\beta}\varphi(m) \qquad \text{for all } m \leq n\ .$$

This was introduced in [DKKT, '03]. They showed that a greedy basis with the URP is bidemocratic, and that a greedy basis (e_i) of a Banach space X with non-trivial type has the URP.

Corollary [DKOSZ] Let $1 < p < \infty$. For all $\varepsilon > 0$ there is an equivalent norm on $L_p[0,1]$ in which the Haar basis is normalized, 1-unconditional, 1-bidemocratic and $(1+\varepsilon)$-greedy.

Problem: Can one make the Haar basis 1-greedy?

Problem: Can one make a bidemocratic, greedy basis 1-greedy?

The general case

Lemma [DKOSZ] Let (e_i) be a normalized, 1-unconditional, Δ-democratic basis of a Banach space X. Given $0 < q < 1$, fix $C > \frac{\Delta}{q(1-q)}$ and set

$$\mathcal{A} = \left\{ A \subset \mathbb{N} : \; n = |A| < \infty, \; \left\| \frac{\varphi(n)}{n} \mathbf{1}_A \right\|^* \leq C \right\} .$$

Then $\forall$ finite $E \subset \mathbb{N}$ there exists $A \in \mathcal{A}$ such that $A \subset E$ and $|A| \geq q|E|$.

Corollary [DKOSZ] Let (e_i) be a greedy basis of a Banach space X. For any $\varepsilon > 0$ there is an equivalent norm on X with respect to which (e_i) is normalized, 1-unconditional and $(1+\varepsilon)$-democratic, and hence $(2+\varepsilon)$-greedy.

Remark: We cannot replace $(1+\varepsilon)$-democratic by 1-democratic. *E.g.*, Tsirelson space or dyadic Hardy space H_1 [DOSZ, '11].

Problem: Can we replace $(2+\varepsilon)$-greedy by $(1+\varepsilon)$-greedy?

Proof of Lemma

Fix $\tau > 0$ such that $C > \frac{(1+\tau)\Delta}{\tau q(1-q)}$. Let $E \subset \mathbb{N}$ and $n = |E|$. Set $F_1 = E$.

Pick $z^{(1)} \in B_{X^*}$ with $\langle \mathbf{1}_{F_1}, z^{(1)} \rangle = \|\mathbf{1}_{F_1}\|$. WLOG $z^{(1)} \geq 0$ and $\text{supp}(z^{(1)}) \subset F_1$. Let $E_1 = \{i \in F_1 : z_i^{(1)} \geq \tau\}$, and $F_2 = F_1 \setminus E_1$ If $|F_2| < (1-q)n$, then stop, else go to next step.

Pick $z^{(2)} \in B_{X^*}$ with $\langle \mathbf{1}_{F_2}, z^{(2)} \rangle = \|\mathbf{1}_{F_2}\|$. WLOG $z^{(2)} \geq 0$ and $\text{supp}(z^{(2)}) \subset F_2$. Let $E_2 = \{i \in F_2 : z_i^{(1)} + z_i^{(2)} \geq \tau\}$, and $F_3 = F_2 \setminus E_2$ If $|F_3| < (1-q)n$, then stop, else go to next step.

Continue inductively. After m steps we end up with disjoint subsets $E_1, \dots, E_m$ of E, and sets $E = F_1 \supset \cdots \supset F_{m+1}$ where $F_k = E \setminus \bigcup_{i=1}^{k-1} E_i$ for $1 \leq k \leq m+1$, and functionals $z^{(1)}, \dots, z^{(m)} \in B^+_{X^*}$ such that $\text{supp}(z^{(k)}) \subset F_k$ and

$$E_k = \{i \in F_k : z_i^{(1)} + \cdots + z_i^{(k)} \geq \tau\} \quad \text{for } k = 1, \dots, m \ .$$

Proof of Lemma (contd.)

$$\left\langle \mathbf{1}_E, z^{(1)} + \cdots + z^{(m)} \right\rangle = \sum_{k=1}^{m} \left\langle \mathbf{1}_{E_k}, z^{(1)} + \cdots + z^{(m)} \right\rangle$$

$$+ \left\langle \mathbf{1}_{F_{m+1}}, z^{(1)} + \cdots + z^{(m)} \right\rangle < (1+\tau)n \ .$$

$$\left\langle \mathbf{1}_E, z^{(1)} + \cdots + z^{(m)} \right\rangle = \sum_{k=1}^{m} \left\langle \mathbf{1}_{F_k}, z^{(k)} \right\rangle \geq m \frac{\varphi\big((1-q)n\big)}{\Delta} \geq m(1-q)\frac{\varphi(n)}{\Delta} \ .$$

So $m \leq \dfrac{(1+\tau)\Delta}{(1-q)} \cdot \dfrac{n}{\varphi(n)}$. Set $A = \bigcup_{k=1}^{m} E_k$. Then $A \subset E$ and $|A| \geq qn$.

Finally, $\|\tau \mathbf{1}_A\|^* \leq \|z^{(1)} + \cdots + z^{(m)}\|^* \leq m$, and hence

$$\left\| \tfrac{\varphi(|A|)}{|A|} \mathbf{1}_A \right\|^* \leq \tfrac{m\varphi(|A|)}{\tau |A|} \leq \cdots \leq C \ . \qquad \text{QED.}$$

The general case

Corollary [DKOSZ] Let (e_i) be a greedy basis of a Banach space X. For any $\varepsilon > 0$ there is an equivalent norm on X with respect to which (e_i) is normalized, 1-unconditional and $(1+\varepsilon)$-democratic, and hence $(2+\varepsilon)$-greedy.

Remark: We cannot replace $(1+\varepsilon)$-democratic by 1-democratic. *E.g.*, Tsirelson space or dyadic Hardy space H_1 [DOSZ, '11].

Problem: Can we replace $(2+\varepsilon)$-greedy by $(1+\varepsilon)$-greedy?

The δ-parameter

Let φ be a fundamental function φ, *i.e.*, $\varphi \colon \mathbb{N} \to \mathbb{R}^+$ is increasing and $\frac{\varphi(n)}{n}$ is decreasing. Define

$$\delta_\varphi(m) = \liminf_{n\to\infty} \frac{\varphi(mn)}{m\varphi(n)} \ ,$$

and

$$\delta(\varphi) = \lim_{m\to\infty} \delta_\varphi(m) \ .$$

Properties of φ imply that $\delta(\varphi) \in [0,1]$.

If (e_i) has the URP, then $\delta(\varphi) = 0$. Indeed,

$$\varphi(mn) \leq C\left(\tfrac{mn}{n}\right)^\beta \varphi(n) \ , \quad \text{and so} \quad \delta_\varphi(m) \leq Cm^{\beta-1} \ .$$

We are interested in the case $\delta(\varphi) > 0$. *E.g.*, if $\varphi(n) \sim n$ (Tsirelson space T, dyadic Hardy space H_1), or if $\varphi(n) \sim \frac{n}{\log n}$ (Schlumprecht space S).

General greedy renorming

Theorem [DKOSZ] Let (e_i) be a greedy basis of a Banach space X with fundamental function φ. Assume that $\delta(\varphi) > 0$. Then for all $\varepsilon > 0$ there is an equivalent norm on X with respect to which (e_i) is normalized, 1-unconditional and $(1+\varepsilon)$-greedy.

Lemma [DKOSZ] Let φ be a fundamental function with $\delta(\varphi) > 0$. Then for all $\varepsilon > 0$ and for all $m \in \mathbb{N}$ there exists a fundamental function $\psi \sim \varphi$ such that $\delta_\psi(m) > \frac{1}{1+\varepsilon}$.

Corollary [DKOSZ] Let X be either H_1 with the Haar basis or T with the unit vector basis. Then for all $\varepsilon > 0$ there is an equivalent norm on X such that the basis is normalized, 1-unconditional and $(1+\varepsilon)$-greedy.

Open Problems

Problem 1 Let (e_i) be a bidemocratic basis of a Banach space X. Does there exist an equivalent norm on X with respect to which (e_i) is 1-greedy?

Problem 2 [AW, '06, Problem 6.2] Let $1 < p < \infty$. Does there exist an equivalent norm on $L_p[0,1]$ with respect to which the Haar basis is 1-greedy?

Problem 3 Let (e_i) be a greedy basis of a Banach space X. Does there exist for any $\varepsilon > 0$ an equivalent norm on X with respect to which the basis is $(1+\varepsilon)$-greedy?