On Poincaré extensions of rational maps.

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Problem

Given a map R in $Rat(\mathbb{C})$ we want to discuss the existence of a extension of R defined in the hyperbolic space $\mathbb{H}^3$. More precisely, a map

$$Ext(R) : \bar{\mathbb{H}}^3 \to \bar{\mathbb{H}}^3$$

which extends R in $\bar{\mathbb{H}}^3$.

Two basic examples.

The map $z \mapsto z^n$

$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{z \mapsto nz} & \mathbb{C} \\ {\scriptstyle exp}\downarrow & & \downarrow{\scriptstyle exp} \\ \mathbb{C}^* & \xrightarrow{z \mapsto z^n} & \mathbb{C}^* \end{array}$$

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The Lattés family.

$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{z \mapsto nz} & \mathbb{C} \\ {\scriptstyle \wp}\downarrow & & \downarrow{\scriptstyle \wp} \\ \mathbb{C} & \xrightarrow{R_n} & \mathbb{C} \end{array}$$

Geometric Extensions

Let S_1 and S_2 be two conformal orbifold structures supported on the Riemann sphere $\bar{\mathbb{C}}$, such that

$$R : S_1 \to S_2$$

is a holomorphic covering. Assume that there exist two Kleinian groups Γ_1 and Γ_2 with components W_1 and W_2 of the discontinuity sets $\Omega(\Gamma_1)$ and $\Omega(\Gamma_2)$ respectively and

$$S_i = W_i / Stab_{W_i}(\Gamma_i).$$

Assume that there exist $\alpha(R) : W_1 \to W_2$ a Möbius map with

$$\begin{array}{ccc} W_1 & \xrightarrow{\alpha(R)} & W_2 \\ \downarrow & & \downarrow \\ S_1 & \xrightarrow{R} & S_2 \end{array}$$

which induces a homomorphism from Γ_1 to Γ_2.
If $M_i := \bar{\mathbb{H}}^3 \cup \Omega(\Gamma_i)/\Gamma_i$. Then $\alpha(R)$ induces a unique Möbius morphism

$$\tilde{R} : M_1 \to M_2.$$

Poincaré extensions

The map $\tilde{R}$ in the previous slide is an extension of $R : S_1 \to S_2$.

Definition

We call $\tilde{R}$ the Poincaré extension of R. Whenever $\tilde{R}$ exists.

Note that the degree is

$$deg(\tilde{R}) = [\Gamma_2 : \alpha(R)\Gamma_1\alpha(R)^{-1}].$$

Hence $deg(R) \leq deg(\tilde{R})$. With equality when

$$Stab_{W_i}(\Gamma_i) = \Gamma_i.$$

Let $\tilde{R}$ be a Poincaré extension of R, such that the discontinuity sets $\Omega(\Gamma_i)$ are connected. Let $\phi_i : \partial M_i \to S_i$ be identification maps. Assume that there is a homeomorphic extension $\Phi_i : M_i \to \bar{\mathbb{H}}^3$. Then the map $\Phi_2 \circ \tilde{R} \circ \Phi_1^{-1}$ is called *geometric* iff satisfies the following conditions.

1. The sets $\Phi_i(M_i \cup \partial M_i)$ are of the form $\bar{\mathbb{H}}^3 \setminus \{\bigcup \gamma_j\}$ where each γ_j is either a geodesic or a family of finitely many geodesic rays with common starting point. There are no more than countably many curves γ_j.

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2. There exist a continuous extension, on all $\mathbb{H}^3$, which maps complementary geodesics to complementary geodesics.

Hence, a geometric extension is an endomorphism of $\mathbb{H}^3$ such that its restriction to $\Phi_1(M_1)$ is a Poincaré extension.

A list of desirable conditions

1. **Geometric.**
2. **Same degree.**
3. **Dynamical.** These are extensions Ext such that $Ext(R^n) = Ext(R)^n$ for $n = 1, 2, \ldots$.
4. **Semigroup Homomorphisms.** A stronger version of the previous property is to find semigroups $\mathcal{S}$, of rational maps, for which there is an extension Ext defined in all $\mathcal{S}$ such that $Ext(R \circ Q) = Ext(R) \circ Ext(Q)$.
5. **Equivariance under Möbius actions.** When defined on saturated sets under the left and right actions of the Möbius group M. This condition can be weaken, and just consider invariant sets under bi-action of subgroups of M.

Equivariance under Möbius actions

Let $A \subset Rat_d(\mathbb{C})$. Assume that there exist a map

$$Ext : A \to End(\bar{\mathbb{H}}^3)$$

such that $Ext(R)$ is an extension of R for every $R \in A$. Then for every pair of maps h, g in Mob we define

$$\widetilde{Ext}(g \circ R \circ h) = \hat{g} \circ Ext(R) \circ \hat{h}$$

where $\hat{g}$ and $\hat{h}$ are the classical Poincaré extensions of the maps g and h, respectively.

If $\widetilde{Ext}$ is a map from the Möbius bi-orbit of A to $End(\bar{\mathbb{H}}^3)$, then we call Ext a *conformally natural extension* of A.

The situation is tricky, even in the case when A consists of a single point R.
Let $\hat{R}$ be a geometric extension of a rational map R, then any rational map on the Möbius bi-orbit of R has a geometric extension. Let us define an extension of Q by the formula $\hat{Q} = \hat{g} \circ \hat{R} \circ \hat{\gamma}$. If there are no elements h_1 and h_2 in $PSL(2, \mathbb{C})$ such that

$$R \circ h_2 = h_1 \circ R, \tag{*}$$

then this extension is conformally natural. Otherwise, we have the following proposition.

Proposition

Let R be a rational map satisfying formula (*) and which has a geometric extension. Then the extension defines a conformally natural extension on the Möbius bi-orbit of R if and only if the Poincaré extensions of the maps h_i*, in formula (*), are Möbius automorphisms of the Möbius manifolds* M_i*.*

Blaschke maps

For the semigroup of Blaschke maps we have the following theorem:

Theorem

Let B be the semigroup of all Blaschke maps, then there exist an extension defined on B that satisfies conditions 1 to 4. This extension is conformally natural with respect of the group of all Möbius maps that leave the unit circle invariant.

Proposition

Let S' be a semigroup of Blaschke maps, then the extension above restricted on S' is conformally natural with respect to all Möbius transformations if and only if S' does not intersect the Möbius bi orbit of maps of the form $z \mapsto z^n$.

Product extension

On $\bar{\mathbb{H}}^3$ there exist a commutative continuous product structure

$$* : \bar{\mathbb{H}}^3 \times \bar{\mathbb{H}}^3 \to \bar{\mathbb{H}}^3$$

$$(x, y) \mapsto x * y$$

Theorem

*The product * satisfies the following conditions:*

1. *If* $x, y \in \mathbb{C}$*, then* $x * y$ *is the usual product in* $\mathbb{C}$.
2. *If* $x \in \mathbb{C}$ *then* $x * y = \gamma_x(y)$ *where* γ_x *is the Poincaré extension of the map* $z \mapsto xz$.
3. *If* $x * y = (1, 0, 0)$ *then* $y = \hat{\tau}(x)$ *where* $\hat{\tau}$ *is the Poincaré extension of the map* $z \mapsto 1/z$.
4. *if* $x = (0, 0, \alpha)$ *and* $y = (0, 0, \beta)$ *then* $x * y = (0, 0, \alpha\beta)$.

In other words, we can say that this product is a conformally natural extension of the complex product.

With the product above we can construct the Product extension:
If $R = \prod \gamma_i$ with γ_i Möbius. Then

$$\hat{R} = \prod_{*} \hat{\gamma}_i.$$

Theorem

1. *Product extension is a left equivariant extension with the same degree.*
2. *There exist only finitely many extensions.*
3. *If S is the semigroup of all maps z^n, then the product extension in S satisfy all five desirable conditions.*