# Expanding Thurston maps and matings

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## PCF branched coverings $f : S^2 \to S^2$

$P_f := \cup_{n>0} f^n\{\text{critical points}\}$ is the post-critical set,
assumption: $|P_f| < \infty$, i.e. $f$ is post critically finite, PCF,
motivation: PCF-rational maps $z^2$, $z^2 - 1$, $\frac{1}{z^2-1}, \dots$

### Combinatorial (or Thurston) equivalence

conjugacy + isotopy rel postcritical set

$f_0, f_1 : S^2 \to S^2$ are isotopic rel $P_{f_0} = P_{f_1}$ if there is a continuous deformation $f_t : S^2 \to S^2$ s.t. $P_{f_t} = P_{f_0} = P_{f_1}$

### Thurston theorem

A non-Lattès $f$ of degree$> 1$ is combinatorially equivalent to a rational map iff $f$ admits no Thurston obstruction.

if $|P_f| > 4$, then $f$ is non-Lattès

# Obstructions

**Thurston theorem**

A non-Lattès $f$ of degree$> 1$ is combinatorially equivalent to a rational map iff $f$ admits no Thurston obstruction.

A Levy obstruction is a simple closed curve $\ell \subset S^2 \setminus P_f$ that is

- essential: both components of $S^2 \setminus \ell$ contain at least 2 points of $P_f$;
- periodic: $f^n : \ell' \to \ell$ is a homeomorphism for $\ell' \subset S^2 \setminus P_f$ isotopic rel $P_f$ to $\ell$.

**Expansion of PCF-rational maps**

$f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ expands the hyperbolic metric of $\hat{\mathbb{C}} \setminus P_f$:

- $f : \hat{\mathbb{C}} \setminus f^{-1}(P_f) \to \hat{\mathbb{C}} \setminus P_f$ is an isometry
- $\hat{\mathbb{C}} \setminus f^{-1}(P_f) \hookrightarrow \hat{\mathbb{C}} \setminus P_f$ is a strict contraction

(we may take the orbifold $(\hat{\mathbb{C}}, P_f, \mathrm{orb}_f)$ instead of $\hat{\mathbb{C}} \setminus P_f$)

## Obstructions

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### Expansion of PCF-rational maps

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If $f$ has a Levy obstruction $\ell_0$, then $f$ admits no expanding metric. (Otherwise, $\ell_0$ would be isotopic to a very short curve via the pullback iteration $f^{-n} : \ell_n \to \ell_{n+1} \sim_{\text{isotopic}} \ell_0$.)
If $f$ is a polynomial or has degree 2, then every Thurston obstruction of $f$ is a Levy obstruction.

## Expanding maps

We say $f : S^2 \to S^2$ is expanding if

- there is a length metric $\mu$ on the orbifold $(S^2, P_f, \mathrm{orb}_f)$ s.t. $f$ expands $\mu$;
- Böttcher normalization: the first return map of $f$ at every periodic critical point is locally conjugate to $z \to z^d$.

As for rational maps:

**Rigidity**

Two expanding maps are equivalent iff they are conjugate.

**Characterization (j.w. Laurent Bartholdi)**

A non-Lattès $f : S^2 \to S^2$ of degree$> 1$ is combinatorially equivalent to an expanding map iff $f$ admits no Levy obstruction.

## Applications

**Iterated monodromy groups**

IMGs of non-Lattès Levy-free maps are contracting

In particular, many symbolic algorithms are available.

# Mating

$f(z) = z^d + a_{d-1}z^{d-1} + \dots$ and $g(z) = z^d + b_{d-1}z^{d-1} + \dots$ are polynomials viewed as maps on $\mathbb{C}$

Formal mating $f \sqcup g : S^2 \to S^2$ is the map s.t.

- $\bar{f}$ acts on the northern hemisphere;
- $g$ acts on the southern hemisphere.

Geometric mating $f \cup_{S^1} g : K_f \cup_{S^1} K_g \to K_f \cup_{S^1} K_g$ is the induced map on the space obtained by gluing filled-in Julia sets of $f$ and $g$.

**Fact**

If $f \sqcup g$ is equivalent to a rational (or expanding) map $f \square g$, then $f \cup_{S^1} g$ : is conjugate to $f \square g$.

This is how it is often established that $K_f \cup_{S^1} K_g$ is a sphere.

## Matings

**Mary Rees, Tan Lei**

Briefly: two quadratic PCF-polynomials are "mateable" iff they are not in conjugate limbs of the Mandelbrot set.

in degree 2 the next theorem is a "part" the previous theorem because every expanding map is conjugate to a rational map

**Higher degree (j.w. Laurent Bartholdi)**

$f, g$ are PCF-hyperbolic polynomials. TFAE:

**(1)** $f \sqcup g$ is equivalent to an expanding map $f \square g$;

**(2)** $K_f \cup_{S^1} K_g$ is a 2-sphere;

**(3)** no finite set of periodic external rays of $f \sqcup g$ disconnect $S^2$.

Tan Lei and Shishikura's example shows that the last theorem is not true for rational maps (if deg>2): there are more complicated than Levy obstructions