Universality in Several Complex Variables

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Birkhoff universality theorem

There exists a universal entire function on $\mathbb{C}$, that is, a function f whose translates approximate all entire functions.

That is, for each entire function g, there is a sequence of complex numbers $a_n, n = 1, 2, \cdots$, such that

$$f(z + a_n) \to g(z).$$

Most entire functions are universal

No example is known!

Voronin's universality theorem states that
the Riemann zeta-function $\zeta(z)$ is universal.

The Riemann zeta-function is not entire.

But almost.

It has only one pole, that pole is simple and the residue is 1.

Voronin universality theorem

All holmorphic functions in the strip

$$S = \{z = x + iy : 1/2 < x < 1\}$$

omitting zero can be approximated by vertical translates $\zeta(z + it)$ of ζ.

More precisely, for each compact $K \subset S$, having connected complement, for each g holomorphic and zero-free on K, for each $\epsilon > 0$, there is a $t \in \mathbb{R}$ such that

$$\max_{z \in K} |\zeta(z + it) - g(z)| < \epsilon.$$

Moreover, the set of such t has positive lower density.

Voronin Universality Theorem
For each $K \subset S$ with connected complement, for each g holomorphic and zero-free on K, for each $\epsilon > 0$,

$$\liminf_{T \nearrow \infty} \frac{1}{T} meas\{t \in [0, T] : \max_{z \in K} |\zeta(z + it) - g(z)| < \epsilon\} > 0$$

Bagchi Theorem The following are equivalent:

1) For each $K \subset S$, for each $\epsilon > 0$,

$$\liminf_{T \nearrow \infty} \frac{1}{T} meas\{t \in [0, T] : \max_{z \in K} |\zeta(z + it) - \zeta(z)| < \epsilon\} > 0$$

2) The Riemann Hypotheisis

K compact

$H(K)$ = holomorphic functions on K

$A(K) = C(K) \cap H(K^o)$.

Problem 1: Zero-free approximation

Theorem[Mergelyan] If $\mathbb{C} \setminus K$ is connected, then, for every $f \in A(K)$ and $\epsilon > 0$, there is a polynomial such that $\max_{z \in K} |p(z) - f(z)| < \epsilon$.

Corollary If $\mathbb{C} \setminus K$ is connected, then, for every $f \in A(K)$ zero-free on K and $\epsilon > 0$, there is a polynomial zero-free on K, such that $\max_{z \in K} |p(z) - f(z)| < \epsilon$.

Conjecture If $\mathbb{C} \setminus K$ is connected, then, for every $f \in A(K)$ zero-free on K^o, and $\epsilon > 0$, there is a polynomial zero-free on K, such that $\max_{z \in K} |p(z) - f(z)| < \epsilon$.

Conjecture[Andersson] If $\mathbb{C} \setminus K$ is connected, and K lies in the strip $1/2 < \Re z < 1$, then, for every $f \in A(K)$ zero-free on K^o, and $\epsilon > 0$, the set of $t \in \mathbb{R}$ such that $\max_{z \in K} |\zeta(z + it) - f(z)| < \epsilon$ is of positive lower density.

Theorem[Andersson] Conjectures 1 and 2 are equivalent!!

Ω domain in $\mathbb{C}^n$

$H(\Omega)$ = holomorphic functions on Ω

$Aut(\Omega)$ = automorphism group of Ω

$f \in H(\Omega)$ is *universal in* $A \subset H(\Omega)$, if

$$\{f \circ \varphi : \varphi \in Aut(\Omega)\} \quad \text{dense in} \quad A$$

1929 Birkhoff. There exists a universal $f \in H(\mathbb{C})$

1941 Seidel-Walsh. There exists a universal $f \in H(\mathbb{D})$

These results extend to $\mathbb{C}^n$ and $\mathbb{B}^n$ easily.

1955 Heins. Exists a Blaschke product universal in

unit ball of $\quad H(\mathbb{D})$

1979 Chee. There exists a universal f in

unit ball of $\quad H(\mathbb{D}^n) \quad$ and $\quad H(\mathbb{B}^n)$

2005 G. & Xiao Jie. Exists a universal *inner* f in

unit ball of $\quad H(\mathbb{B}^n)$

2007 Aron & Gorkin. Many universal f in

unit ball of $\quad H(\mathbb{B}^n)$

2007 Bayart & Gorkin. Not just $\mathbb{B}^n$

2008 Gorkin & León-Saavedra & Mortini. Characterize such universal functions

Approximation by bounded holomorphic functions

2009 G and Mel'nikov Suppose $\Omega \subset \overline{\mathbb{C}}$ open.
The following are equivalent:

(i) for each $f \in H(\Omega)$, there are $g_j \in H^\infty(\Omega)$, $g_j \to f$.

(ii) for each open $U \subset \overline{\mathbb{C}}$, with $\partial U \subset \Omega$,

$$U \setminus \Omega \neq \emptyset \quad \Rightarrow \quad \gamma(U \setminus \Omega) > 0. \tag{1}$$

(iii) for each open $U \subset \overline{\mathbb{C}}$, (1) holds.

Here, $\gamma(E)$ = analytic capacity. Sets of analytic capacity zero are precisely removable sets for bounded holomorphic functions.

Potential theoretic analogue

2008 Sylvain Roy. Suppose $\Omega \subset \mathbb{R}^n$ is Greenian.
The following are equivalent:

(i) for each $u \in S(\Omega)$, there are $v_j \in S(\Omega)$ upper bounded, $v_j \searrow u$ and $v_j = u$ eventually on compacta.

(ii) for each bounded open $U \subset \mathbb{R}^n$, with $\partial U \subset \Omega$, and also each open $U \subset \overline{\mathbb{C}}$, with $\partial U \subset \Omega$, if $n = 2$,

$$U \setminus \Omega \neq \emptyset \quad \Rightarrow \quad c(U \setminus \Omega) > 0.$$

Here, $c(E)$ = capacity. Sets of capacity zero are precisely removable sets for bounded harmonic functions.

Greenian domains are precisely those admitting non-constant upper bounded subharmonic functions.

Universal series in $\mathbb{C}^n, n > 1$

Homogeneous expansion f holomorphic at $0 \in \mathbb{C}^n$

$$f(z) = \sum_{j=0}^{\infty} h_j(z), \quad |z| < r,$$

where h_j homogeneous polynomials.

Theorem For $r \geq 0$, there exists a homogeneous series which converges for $|z| < r$ and for each compact convex K outside $|z| \leq r$, and each polynomial p and each $\epsilon > 0$, there is a J such that

$$\sup_{z \in K} \left| \sum_{j=0}^{J} h_j(z) - p(z) \right| < \epsilon.$$

This theorem is due to:

R. Clouätre when $r = 0$. Bull. CMS, 2011.

N. J. Daras; V. Nestoridis when $r > 0$.
ArXiv, February 2013.

Universal Plurisubharmonic Functions

Ω open set in $\mathbb{R}^n$. A continuous function $u : \Omega \to \mathbb{R}$ is **subharmonic** iff for each closed ball $\overline{B} \subset \Omega$ and function h continuous on $\overline{B}$ and harmonic on B,

$$u \le h \quad \text{on} \quad \partial B \qquad \Rightarrow \qquad u \le h \quad \text{on} \quad B.$$

Ω open set in $\mathbb{C}^n$. A continuous function $u : \Omega \to \mathbb{R}$ is **plurisubharmonic** iff for each closed complex disc $\overline{D} \subset \Omega$ and function h continuous on $\overline{D}$ and harmonic on D,

$$u \le h \quad \text{on} \quad \partial D \qquad \Rightarrow \qquad u \le h \quad \text{on} \quad D.$$

For $\mathbb{F} = \mathbb{R}$ (respectively $\mathbb{F} = \mathbb{C}$),
$S_c(\mathbb{F}^n) =$
cont. subharm. (respectively, plurisubharm.) functions on $\mathbb{R}^n$ (resp, on $\mathbb{C}^n$).

2007 G. and Pouryayevali.
There exists a universal $f \in S_c(\mathbb{F}^n)$.
Namely, its translates $f(z+a), a \in \mathbb{F}^n$, are dense in $S_c(F^n)$.

$\Omega \subset \mathbb{R}^n$. $u : \Omega \to [-\infty, +\infty)$ uppersemicontinuous is **subharmonic** iff for each ball $\overline{B} \subset \Omega$ and h continuous on $\overline{B}$ and harmonic on B,

$$\limsup_{x \in B, x \to y} u(x) \le h(y) \quad \forall y \in \partial B \quad \Rightarrow \quad u \le h \quad \text{on} \quad B.$$

$\Omega \subset \mathbb{C}^n$. $u : \Omega \to [-\infty, +\infty)$ uppersemicontinuous is **plurisubharmonic** iff for each complex disc $\overline{D} \subset \Omega$ and function h continuous on $\overline{D}$ and harmonic on D,

$$\limsup_{z \in D, z \to \zeta} u(z) \le h(\zeta) \quad \forall \zeta \in \partial D \quad \Rightarrow \quad u \le h \quad \text{on} \quad D.$$

For sequence $\{u_j\}$ in $S(\mathbb{F}^n)$ and $u \in S(\mathbb{F}^n)$, we write

$$u_j \searrow u$$

if $u_j \to u$ pointwise and for each compact $K \subset \mathbb{F}^n$, there is a j_K such that,

$$u_j(x) \ge u_{j+1}(x), \quad \forall j \ge j_K \quad \text{and} \quad x \in K.$$

2007 G and Pouryayevali. There is $u \in S_c(\mathbb{F}^n)$ universal in $S(\mathbb{F}^n)$: for each $v \in S(\mathbb{F}^n)$, there is a sequence $a_j \in \mathbb{F}^n$, such that $u(\cdot + a_j) \searrow v(\cdot)$.

Universal functions on arbitrary Stein manifolds

X Stein manifold. $\mathbb{B}^n$ ball in $\mathbb{C}^n$. φ biholomorphic mapping of neighborhood of $\overline{\mathbb{B}}^n$ into X.
$B = \varphi(\mathbb{B}^n)$ is a parametric ball in X.
Let $\Phi(X)$ be the family of all φ such that
$\varphi : \mathbb{B}^n \to X$ is a parametric ball in X.
$f \in H(X)$ **universal** with respect to $H(\mathbb{B}^n)$ if the family $f \circ \Phi(X)$ dense in $H(\mathbb{B}^n)$.

2005 G and Pouryayevali
For each Stein manifold X of dimension n,
most $f \in H(X)$ are universal with respect to $H(\mathbb{B}^n)$.

We use a general Rouché Theorem.

Rouché type theorem

If $G \subset \mathbb{R}^m$ is a bounded domain, $\overline{G}$ is called a *compact domain.*

$f : \overline{G} \to \mathbb{R}^m$ continuous mapping.

$$y \in \mathbb{R}^m, \quad y \notin f(\partial G)$$

Topological index or *degree*

$$\mu(y, f, \Omega)$$

of the mapping f at the point y.

This ia the main topological notion for an equidimensional mapping of a domain Ω.

Theorem.(G and Pouryayevali) $\overline{\Omega}$ compact domain in an m-dimensional orientable real manifold X and let f and h be continuous mappings $\overline{\Omega} \to \mathbb{R}^m$ such that $|h| < |f|$ on $\partial\Omega$. If $0 \notin f(\partial\Omega)$, then

$$\mu(0, f, \overline{\Omega}) = \mu(0, f + h, \overline{\Omega}).$$

Problem 2

Suppose $u : \mathbb{C}^n \to [-\infty, +\infty)$ and

$u|L$ is subharmonic, for each complex line L.

Is u plurisubharmonic?

Problem 3

Is every plurisubharmonic function subharmonic?

On Kähler manifolds, it seems the answer is YES.

What about an arbitrary manifold?

My recent papers on universality in SCV or the zeta-function

MR3170746 Pending Andersson, J.; Gauthier, P. M. Mergelyan's theorem with polynomials non-vanishing on unions of sets. Complex Var. Elliptic Equ. 59 (2014), no. 1, 99-109.

MR3078223 Reviewed Gauthier, P. M. Universally over-convergent power series via the Riemann zeta-function. Canad. Math. Bull. 56 (2013), no. 3, 544-550.

MR3052286 Reviewed Gauthier, P. M. Approximating the Riemann zeta-function by strongly recurrent functions. Blaschke products and their applications, 31-42, Fields Inst. Commun., 65, Springer, New York, 2013.

MR3113295 Pending Gauthier, Paul M.; Knese, Greg Zero-free polynomial approximation on a chain of Jordan domains. Ann. Sci. Math. QuÃ©bec 36 (2012), no. 1, 107-112 (2013).

MR3058520 Reviewed Gauthier, Paul M. Approximating all meromorphic functions by linear motions of the Riemann zeta-function. Comput. Methods Funct. Theory 12 (2012), no. 2, 517-526.

MR3051813 Reviewed Donzelli, F.; Gauthier, P. M. On the instability of the Riemann hypothesis for varieties over finite fields. Izv. Nats. Akad. Nauk Armenii Mat. 47 (2012), no. 3, 55–66; translation in J. Contemp. Math. Anal. 47 (2012), no. 3, 124-133

MR2977292 Reviewed Gauthier, Paul M. Approximating functions by the Riemann zeta-function and by polynomials with zero constraints. Comput. Methods Funct. Theory 12 (2012), no. 1, 257-271.

MR2885421 Reviewed Gauthier, P. M.; Tarkhanov, N. On the instability of the Riemann hypothesis for curves over finite fields. J. Approx. Theory 164 (2012), no. 4, 504-515.

MR2791327 Reviewed Gauthier, Paul M. Approximation of and by the Riemann zeta-function. Comput. Methods Funct. Theory 10 (2010), no. 2, 603-638.

MR2648984 Reviewed Gauthier, Paul M.; Xarles, Xavier Perturbations of L-functions with or without non-trivial zeros off the critical line. New directions in value-distribution theory of zeta and L-functions, 65-83, Ber. Math., Shaker Verlag, Aachen, 2009.

MR2500510 Reviewed Gauthier, Paul M.; Melnikov, Mark S. Compact approximation by bounded functions and functions continuous up to the boundary. Linear and complex analysis, 61-66, Amer. Math. Soc. Transl. Ser. 2, 226, Amer. Math. Soc., Providence, RI, 200

DZIENKUYE!

THANKYOU!

MERCI!