Wandering domains in Eremenko-Lyubich's class and Bishop's example

N. Fagella, S. Godillon X. Jarque (Universitat de Barcelona)

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Introduction

Let $f : \mathbb{C} \to \mathbb{C}$ a transcendental entire function.

The Julia set of f, $\mathcal{J}(f)$, is the set of points $z \in \mathbb{C}$ for which the family of iterates $\{f^n(z)\}_{n\in\mathbb{N}}$ fails to be a normal family at z. The Fatou set, $\mathcal{F}(f)$, is defined as $\mathbb{C} \setminus \mathcal{J}(f)$. Each connected component of $\mathcal{F}(f)$ is called a Fatou domain.

Each Fatou domain is mapped into another Fatou domain. So, they can be either (eventually) periodic or **wandering**. In this later case we have

$$f^n(U) \cap f^m(U) = \emptyset \quad \forall n \neq m, \quad n, m \in \mathbb{Z}.$$

Introduction

Theorem (Sullivan 1985): Let $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a rational map and let U be a Fatou domain of f. Then $f^{\ell}(U)$ is eventually periodic for some $\ell \geq 0$.
In other words, rational functions do not have wandering domains.

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Definition: The singular set, $S(f)$ is the set of (finite) singularities of f^{-1} (critical values, asymptotic values and limits of those).

We say that f is critically finite (Speiser class, class $\mathcal{S}$) if $S(f)$ is finite.
We say that f is bounded type (Eremenko-Lyubich class, class $\mathcal{B}$) if $S(f)$ is bounded.

Eremenko-Lyubich and Golberg-Keen (1987-8): Sullivan's Theorem extends to transcendental entire function in class $\mathcal{S}$.

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Theorem (Baker and Töpfer): If $U \subset \mathcal{F}(f)$ is multiply connected then

(a) $f^n|_U \mapsto \infty$ uniformly on compact subset of U, (maximum principle)

(b) U is bounded, and

(c) U is a wandering domain.

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Let $g(z) = \frac{1}{4e}\, z^2 \prod_{n=1}^{\infty} \left(1 + \frac{z}{a_n}\right)$, with $1 < a_1 < a_2 < \ldots$.

Theorem (Baker's example, 70's): If the sequence $\{a_j\}_{j \geq 0}$ is appropriately chosen then g has a (Baker)-wandering (multiply connected) domain.

W. Bergweiler, M. Kisaka, P. Rippon, M. Shishikura, G. Stallard...

Constant limit functions

Theorem (Fatou 1920): Let U a wandering domain of f. All limit functions of the sequences $\{f^{n_k}|_U\}$ are constant.

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- $\{f^n|_U\} \to \infty$ (uniformly in compact subsets of U)
- $\{f^{n_k}|_U\} \to \infty$ and $\{f^{m_k}|_U\} \to a \in \mathcal{J}(f) \subset \mathbb{C}$ (oscillating)
- If $\{f^{n_k}|_U\} \to a$ then $a \neq \infty$ (bounded)

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Remark: Baker's multiply connected example is of first type.

Theorem (Eremenko-Lyubich, 1987): There exits an entire function which has an oscillating wandering component U (with infinitely many finite constant limit points).

Remark: (As far as I know) there are no examples of the third type.

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Let f be an entire map, and let $E = \bigcup_{s \in \mathcal{S}(f)} \bigcup_{n \geq 0} \{f^n(s)\}$

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Theorem (Baker, 1976): Let U be a wandering domain. Then all constant limit functions of $\{f^{n_k}|_U\}$ belong to $L := \overline{E} \cup \infty$.

Theorem (BHKMT, 1993): Let U a wandering domain. Then all constant limit functions of $\{f^{n_k}|_U\}$ belong to $E' \cup \infty$, where E' is the set of finite limit points of E.

Existence of wandering domains in class $\mathcal{B}$

Theorem (Eremenko-Lyubich 1985): Let $f \in \mathcal{B}$. If U wanders then either it oscillates or it has bounded orbit. That is,

$$\{f^n|_U\} \nrightarrow \infty. \quad \text{(locally uniformly)}$$

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Question (Mihaljević-Brand and Rempe-Gillen): Let $f \in \mathcal{B}$. Assume

$$\lim_{n\to\infty} \inf_{s\in\mathcal{S}(f)} |f^n(s)| = \infty.$$

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Can f have wandering domains?

Theorem (Mihaljević-Brandt and Rempe-Guillen, 2013): The answer is NO, if f satisfies a further condition.

Existence of wandering domains in class $\mathcal{B}$

Key Lemma (Mihaljević-Brand and Rempe-Gillen): Let U be a hyperbolic domain in $\mathbb{C}$.

- Let $U' \subset U$ be open and let $f : U' \to U$ be a holomorphic covering map.
- Assume that there is an open connected set $W \subset U'$ such that $f^n(W) \subset U', \; n \geq 0$. The wandering domain...
- Let $D \subset U$ be open and set $V := f^{-1}(D)$. Let $w \in W$. Suppose $\{n_k\}$ is a sequence such that

$$f^{n_k}(w) \in D \quad \text{and} \quad d_U(f^{n_k}(w), U \setminus D) \to \infty$$

Then

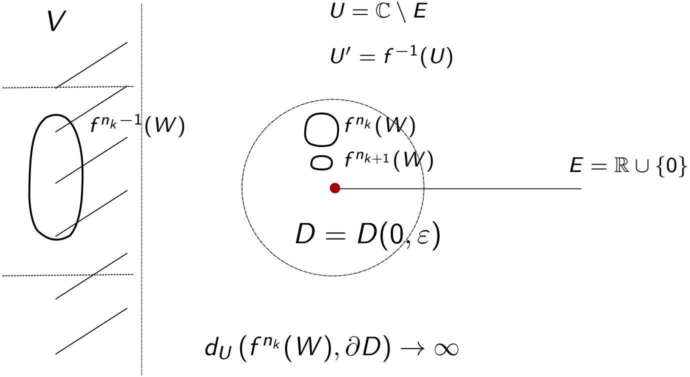
$$d_U(f^{n_k - 1}(w), U \setminus V) \to \infty$$

Existence of wandering domains in class $\mathcal{B}$

$$f(z) = \exp(z)$$

$$U = \mathbb{C} \setminus E$$

$$U' = f^{-1}(U)$$



$$d_U\left(f^{n_k}(W), \partial D\right) \to \infty$$

$$d_U\left(f^{n_k-1}(W), \partial V\right) < \infty!!$$

Existence of wandering domains in class $\mathcal{B}$

Theorem (Bishop, 2012-3): There exists $f \in \mathcal{B}$ such that f has two grand orbits of oscillating wandering domains. (This result is a nice, non-trivial application of a much more general result of C. Bishop on quasiconformal foldings)

Remark: Bishop's example is not a negative answer to Mihaljević-Brandt and Rempe-Gillen question since f has oscillating critical orbits.

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Theorem A (Godillon, Fagella, J., 2014): Let f be given by Bishop's example. Then f has **exactly two** grand orbits of wandering domains. (no unexpected wandering domains in Bishop's example)

Composite functions and dynamics

Theorem (Bergweiler and Wang 1998): Let f and g transcendental entire and let $z \in \mathbb{C}$. Then $z \in \mathcal{J}(f \circ g) \iff g(z) \in \mathcal{J}(g \circ f)$.
Assume U is a component of $\mathcal{F}(f \circ g)$ and V is the component of $\mathcal{F}(g \circ f)$ that contains $g(U)$. Then U is a wandering domain $\iff$ V is wandering.

Theorem (Singh 2003): There exist two transcendental entire functions f and g and a domain $U \subset \mathbb{C}$ such that U lies in periodic Fatou components of f, g and $g \circ f$ but lies in a wandering domain of $f \circ g$.

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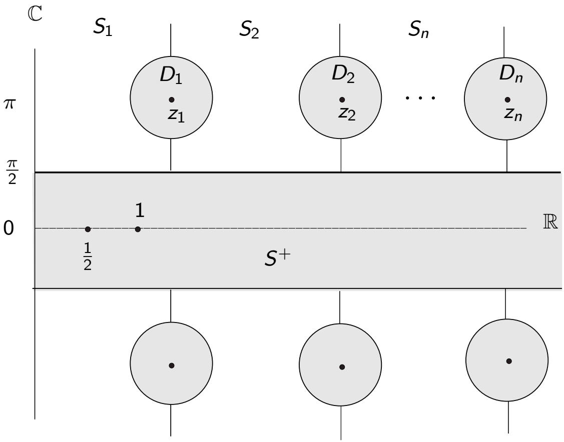
Theorem B (Godillon, Fagella, J., 2014): There exist two transcendental entire functions f and g in class $\mathcal{B}$ such that $f \circ g$ and $g \circ f$ have two grand orbits of wandering domains even though all Fatou domains of f and g are preperiodic.

Sketch of the proof of Theorem A



Remark: $S(f) = \{-1, +1, \cup_{j \geq 1} f(z_j)\}$

$$f(\overline{z}) = \overline{f(z)} \quad \text{and} \quad f(-z) = -f(z)$$

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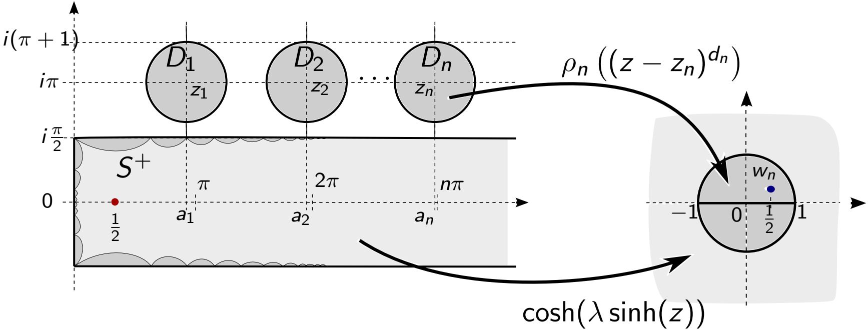
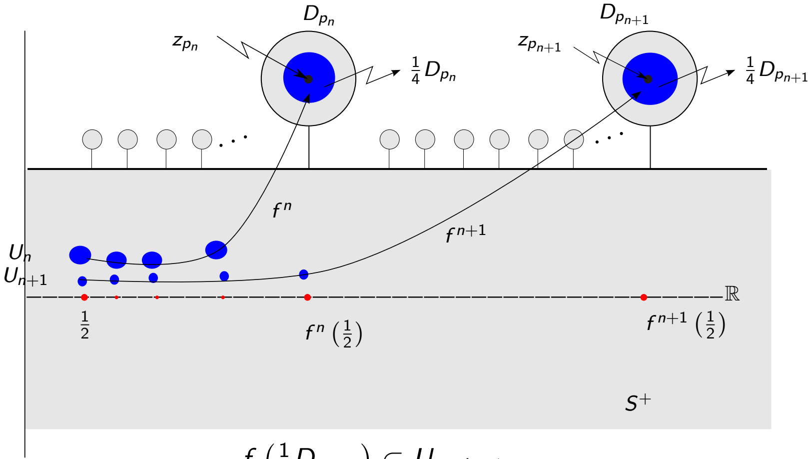


$$f(z) = \begin{cases} \cosh(\lambda \sinh(\phi(z))) & \text{if} \quad \phi(z) \in S^+ \\ \rho_n\left((\phi(z) - z_n)^{d_n}\right) & \text{if} \quad \phi(z) \in D_n \end{cases}$$

Remark: $\phi(z)$ is *the* quasi-conformal integrator, it is arbitrarily close to the identity with $\phi(\mathbb{R}) = \mathbb{R}$, $\phi(0) = 0$, $\phi(\infty) = \infty$ and $\phi'(\infty) = 1$.

Sketch of the proof of Theorem A: $(\lambda, \{d_n\}, \{w_n\})$



$$f\left(\tfrac{1}{4}D_{p_{n+k}}\right) \subset U_{n+k+1}$$

$f(z_{p_n}) = w_{n+1} \in U_{n+1} \to \frac{1}{2}$ $\qquad$ $f(z_\ell) = \frac{1}{2}, \ell \neq p_n$

Sketch of the proof of Theorem A: Location

Lemma 1: Let W be a wandering domain of f. Then there exist two subsequences of positive integers $(n_k)_{k\geqslant 1}$ and $(m_k)_{k\geqslant 1}$ such that $f^{n_k}|_W \to \frac{1}{2}$ as $k \to +\infty$ and $f^{n_k-1}(W) \subset D_{m_k}$ for every $k \geqslant 1$.

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Proof: (i) $E' = \bigcup_{n\geq 0} f^n\left(\frac{1}{2}\right)$, so $f^{n_k}|_W \to a := f^{\ell}\left(\frac{1}{2}\right)$.

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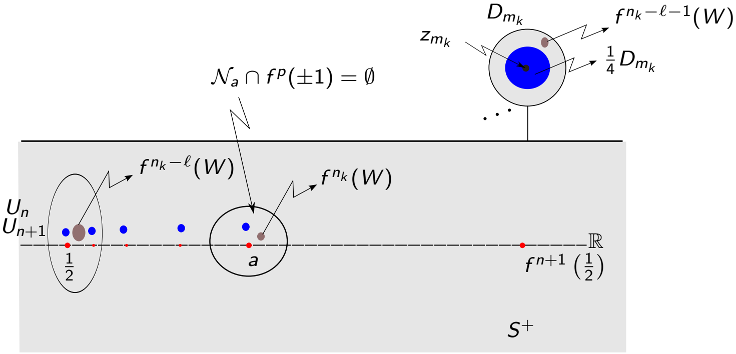


Sketch of the proof of Theorem A: Hyperbolic Lemma

Lemma 2: For every $n \geqslant 1$, assume that either $w_n = \frac{1}{2}$ or $f(\frac{1}{4}D_n) \subset \cup_{m\geqslant 1} U_m$. Let $(n_k)_{k\geqslant 1}$ be as before for a wandering domain W, and let A be the following closed set

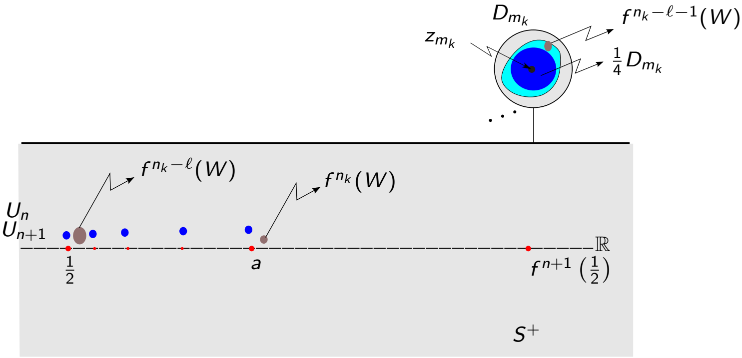
$$A := \big(-\infty, -\frac{1}{2}\big] \cup \big[\frac{1}{2}, +\infty\big) \cup \bigcup_{n\geqslant 1} \bigcup_{k=0}^{n} \overline{f^k(U_n)}.$$

If $f^n(W) \cap A = \emptyset$ for every $n \geqslant 0$ and $U := \mathbb{C} \setminus A$, then

$$\mathrm{dist}_U \left(f^{n_k-1}(W), U \setminus f^{-1}(\mathbb{D})\right) \to +\infty \text{ as } k \to +\infty,$$

Proof: It follows from the Key Lemma in Mihaljević-Brand and Rempe-Gillen.

Sketch of the proof of Theorem A: Final argument



$$\mathrm{dist}_U\left(f^{n_k-1}(W), U \setminus f^{-1}(\mathbb{D})\right) \to +\infty \text{ as } k \to +\infty!!$$

dziękuję

Moltes gràcies

Theorem: There exist two transcendental entire functions f and g in class $\mathcal{B}$, with corresponding sequences of euclidean disks $(U_n^f)_{n\geqslant 1}$ and $(U_n^g)_{n\geqslant 1}$ respectively and same subsequence of positive integers $(p_n)_{n\geqslant 1}$, such that for every $n \geqslant N$ large enough $U_n := U_n^f \cap U_n^g$ is not empty and

$$\begin{cases} f^{4n+1}(U_{4n}\quad) & \subset & U_{4n} \\ f^{4n+2}(U_{4n+1}) & \subset & U_{4n+1} \\ f^{4n+3}(U_{4n+2}) & \subset & U_{4n+3} \\ f^{4n+4}(U_{4n+3}) & \subset & U_{4n+4} \end{cases}, \quad \begin{cases} g^{4n+1}(U_{4n}\quad) & \subset & U_{4n+1} \\ g^{4n+2}(U_{4n+1}) & \subset & U_{4n+2} \\ g^{4n+3}(U_{4n+2}) & \subset & U_{4n+2} \\ g^{4n+4}(U_{4n+3}) & \subset & U_{4n+3} \end{cases},$$

$$(f\circ g)^{8n+5}(U_{4n}) \subset U_{4n+4}, \quad \text{and} \quad \begin{cases} (g\circ f)^{2n}(U_{4n}) \subset f^{-1}(U_{4n}) \\ (g\circ f)^{8n+5}\left(f^{-1}(U_{4n})\right) \subset f^{-1}(U_{4n+4}) \end{cases}$$

In particular, the open sets $(U_n)_{n\geqslant 4N}$ lie in periodic Fatou domains for f and g, but lie in wandering Fatou domains for $f\circ g$ and $g\circ f$.