Local and global finite branching of solutions of ODEs in the complex plane

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Painlevé property and Painlevé equations

1900's, Painlevé, Gambier, Fuchs: Classification of all second-order rational equations

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$$y'' = 6y^2 + z, \qquad y'' = \frac{(y')^2}{y} - \frac{y'}{z} + \frac{1}{z}\left(\alpha y^2 + \beta\right) + \gamma y^3 + \frac{\delta}{y}$$

$$y'' = 2y^3 + zy + \alpha, \quad y'' = \frac{(y')^2}{2y} + \frac{3}{2}y^3 + 4zy^2 + 2(z^2 - \alpha)y + \frac{\beta}{y}$$

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Solutions of P_I, P_{II} and P_{IV} are meromorphic in the whole of $\mathbb{C}$. (Hinkkanen & Laine 1999, Steinmetz 2000)

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Theorem (Malmquist 1920)

Suppose the first-order algebraic equation with rational coefficients,

$$F(z, y, y') = 0$$

admits a transcendental meromorphic solution. Then it reduces, by a transformation $w = R(z, y, y')$*, either to a Riccati equation,*

$$w' = a(z)w^2 + b(z)w + c(z),$$

or to the equation satisfied by the Weierstraß $\wp$ *function,*

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Remark

This result was generalised by Eremenko in a 1982 paper to the case of admissible solutions using Nevanlinna theory.

Admissible solutions

We adopt the usual notation for the Nevanlinna characteristic

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Definition

Suppose $y(z)$ is meromorphic and satisfies a differential equation

$$F\left(z, y, y', \dots, y^{(p)}\right) = 0,$$

with coefficients $\{a_\lambda, \lambda \in I\}$. Then y is called admissible solution if

$$T(r, a_\lambda) = S(r, y) \quad \forall \lambda \in I.$$

Equations with movable algebraic singularities

The following authors have studied classes of 2^{nd}-order differential equations for which all movable singularities in the complex plane of all solutions are either poles or algebraic branch points.

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$$y'' = E(z, y)(y')^2 + F(z, y)y' + G(z, y),$$

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$$y'' = E(z, y)(y')^2 + F(z, y)y' + G(z, y),$$

All solutions to these equations are locally finite branched - **but in general not globally!**

Global finite branching: Algebroid functions

Definition

Suppose $y(z)$ satisfies an irreducible algebraic equation

$$y^n + s_1(z)y^{n-1} + \cdots + s_n(z) = 0.$$

Then y is called an n-valued algebroid function. If all functions $s_1, \ldots, s_n$ are rational, then y is called algebraic, otherwise transcendental algebroid.

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Locally, an algebroid function is represented by series expansions

$$y_i(z) = \sum_{j=m_i}^{\infty} c_j (z - z_0)^{j/n_i}, \quad c_j \in \mathbb{C}, \quad m_i \in \mathbb{Z}, \quad n_i \in \mathbb{N}.$$

$i = 1, \ldots, k$, where $n_1 + \cdots + n_k = n$.

First-order equations with algebroid solutions

Theorem (Malmquist 1913)

Suppose the first-order rational equation

$$y' = \frac{P(z, y)}{Q(z, y)} \tag{1}$$

has a transcendental algebroid solution. Then the equation can be reduced, by a transformation $w = R(z, y)$, to a Riccati equation.

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Remark

Usually, Malmquist's theorem is cited as the following: If equation (1) has a transcendental meromorphic solution, then it is already a Riccati equation. In this form it was also proved by Yosida (1933) using Nevanlinna theory.

Second-order equations with algebroid solutions

Under the assumption of an admissible algebroid solution, to what types can equations in certain classes of second-order equations be reduced to?

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Example

Consider a second-order equation in the polynomial class,

$$y'' = y^5 + a_4(z)y^4 + a_3(z)y^3 + a_2(z)y^2 + a_1(z)y + a_0(z),$$

and suppose their exists a 2-valued algebroid solution y satisfying

$$y^2 + f(z)y + g(z) = 0.$$

Using Nevanlinna theory one can show that

$$T(r,f) = S(r,y), \quad \left(\text{in fact, } f = \frac{2}{5}a_4\right)$$

linear transformation in $y \implies f(z) \mapsto 0$

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Remaining equation for g:

$$g'' = \frac{(g')^2}{2g} + \frac{3}{2}g^3 - 2a_3(z)g^2 + 2a_1(z)g$$

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At any pole z_0 of $g(z)$,

$$g(z) \sim \alpha(z - z_0)^p \quad \Longrightarrow \quad p = -1, \alpha = \pm 1.$$

At a pole z_0 with residue α the following relation is satisfied

$$\alpha a_3''(z_0) + a_3(z_0)a_3'(z_0) - 2a_1'(z_0) = 0.$$

Cases with different types of poles

Nevanlinna theory: $m(r, g) = S(r, g)$, i.e. $N(r, g) \asymp T(r, g)$.
Denote by $N_\alpha(r, g)$ the counting function for the poles of g with residue $\alpha = \pm 1$. Two cases are possible:

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- $N_\alpha(r,g) \asymp T(r,g)$ but $N_{-\alpha}(r,g) = S(r,g)$, then we have $a_1(z) = \frac{\alpha}{2} a_3'(z) + \frac{1}{4} a_3(z)^2 + C$ and g satisfies

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- $N_1(r,g) \asymp T(r,g)$ and $N_{-1}(r,g) \asymp T(r,g)$. Then we have

$$a_3''(z) = 0, \quad \text{and} \quad (a_3^2 - 4a_1)' = 0$$

$\Longrightarrow a_3(z) = -2(az+b)$ and $a_1(z) = (az+b)^2 - c$.

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  $\Longrightarrow a_3(z) = -2(az+b)$ and $a_1(z) = (az+b)^2 - c$.
  Thus if $a = 0$ the equation for g reduces to

$$g'' = \frac{(g')^2}{2g} + \frac{3}{2} g^3 + 4bg^2 + 2(b^2 - c)g,$$

  which is solved in terms of elliptic functions.

- Or, if $a \neq 0$, make a linear transformation in z to set $a = 1$ and $b = 0$ to obtain

$$g'' = \frac{(g')^2}{2g} + \frac{3}{2}g^3 + 4zg^2 + 2(z^2 - c)g,$$

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Thus we have shown that the solutions to the equation

$$y'' = y^5 + a_4(z)y^4 + a_3(z)y^3 + a_2(z)y^2 + a_1(z)y + a_0(z),$$

under the existence of an admissible 2-valued algebroid solution, can be given by solving a Riccati equation, in terms of elliptic functions, or in terms of the fourth Painlevé transcendents.

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Thank you!