Boundary Behavior of Universal Taylor Series
Perspectives of Modern Complex Analysis
Bedlewo, Poland, July 2014

Stephen J. Gardiner and Dmitry Khavinson
dkhavins@usf.edu
http://shell.cas.usf.edu/ dkhavins/

University College Dublin, University of South Florida

July 24, 2014

Outline

Outline

Outline

Outline

Philosophy: An object is called *universal* if, via a countable number of steps it can approximate any object in some "universe".

Philosophy: An object is called *universal* if, via a countable number of steps it can approximate any object in some "universe".

- M. Fekete (1886 - 1957), in a paper written by J. Pal in 1914 constructed $U := \sum_1^\infty a_n x^n$, $a_n \in \mathbb{R}$, ROC $= 0$ such that $\forall g \in C_{\mathbb{R}}[-1,1]$, $g(0) = 0, \exists$ a subsequence of partial sums s_{n_k} of U s.t. $s_{n_k} \rightrightarrows g$ on $[-1,1]$.

Philosophy: An object is called *universal* if, via a countable number of steps it can approximate any object in some "universe".

- M. Fekete (1886 - 1957), in a paper written by J. Pal in 1914 constructed $U := \sum_1^\infty a_n x^n$, $a_n \in \mathbb{R}$, ROC $= 0$ such that $\forall g \in C_{\mathbb{R}}[-1,1]$, $g(0) = 0, \exists$ a subsequence of partial sums s_{n_k} of U s.t. $s_{n_k} \rightrightarrows g$ on $[-1,1]$.

- George D. Birkhoff (1884 - 1944) showed (1929) existence of an entire function $f(z)$ whose translates $f(z+n), n \in \mathbb{N}$ can approximate any entire function uniformly on compact subsets of $\mathbb{C}$.

- G. R. MacLane (1952) constructed an entire function with universal derivatives.

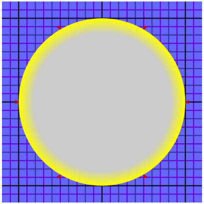
- G. R. MacLane (1952) constructed an entire function with universal derivatives.
- A. I. Seleznev (1952) constructed a power series $\sum a_n z^n$, with ROC $= 0 : \forall$ compact $K \subset \mathbb{C} \setminus \{0\}$: $\mathbb{C} \setminus K$ is connected, $\forall g \in A(K) \, \exists s_{n_k} := \sum_1^{n_k} a_n z^n$, a subsequence of partial sums, $s_{n_k} \rightrightarrows g$ on K.

- G. R. MacLane (1952) constructed an entire function with universal derivatives.
- A. I. Seleznev (1952) constructed a power series $\sum a_n z^n$, with ROC $= 0 : \forall$ compact $K \subset \mathbb{C} \setminus \{0\}$: $\mathbb{C} \setminus K$ is connected, $\forall g \in A(K) \, \exists s_{n_k} := \sum_1^{n_k} a_n z^n$, a subsequence of partial sums, $s_{n_k} \rightrightarrows g$ on K.
- Luh (1970), C. K. Chui and M. N. Parnes (1971) constructed power series with a ROC $r > 0$ that has the above approximation property on $\{|z| > r\}$.

- G. R. MacLane (1952) constructed an entire function with universal derivatives.
- A. I. Seleznev (1952) constructed a power series $\sum a_n z^n$, with ROC $= 0 : \forall$ compact $K \subset \mathbb{C} \setminus \{0\}$: $\mathbb{C} \setminus K$ is connected, $\forall g \in A(K) \, \exists s_{n_k} := \sum_1^{n_k} a_n z^n$, a subsequence of partial sums, $s_{n_k} \rightrightarrows g$ on K.
- Luh (1970), C. K. Chui and M. N. Parnes (1971) constructed power series with a ROC $r > 0$ that has the above approximation property on $\{|z| > r\}$.
- V. Nestorides (1990) proved that universality holds on $\{|z| \geq r\}$.

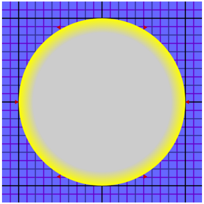
- G. R. MacLane (1952) constructed an entire function with universal derivatives.
- A. I. Seleznev (1952) constructed a power series $\sum a_n z^n$, with ROC $= 0 : \forall$ compact $K \subset \mathbb{C} \setminus \{0\}$: $\mathbb{C} \setminus K$ is connected, $\forall g \in A(K) \, \exists s_{n_k} := \sum_1^{n_k} a_n z^n$, a subsequence of partial sums, $s_{n_k} \rightrightarrows g$ on K.
- Luh (1970), C. K. Chui and M. N. Parnes (1971) constructed power series with a ROC $r > 0$ that has the above approximation property on $\{|z| > r\}$.
- V. Nestorides (1990) proved that universality holds on $\{|z| \geq r\}$. He also showed that the set of universal power series $\mathcal{U}$ is a dense G_δ subset of the space of all holomorphic functions on the disk endowed with the topology of uniform convergence on compact subsets.

Place of Action

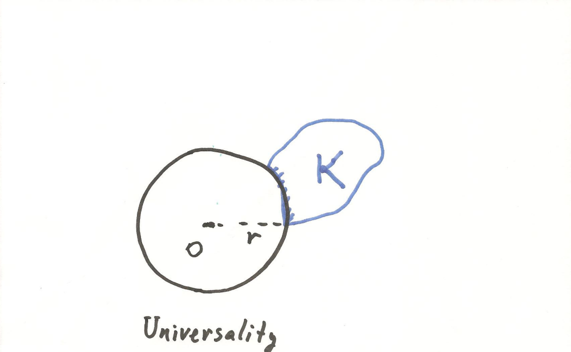


open disk

Place of Action



open disk



The Main Result

The Main Result

Theorem 1

Let $\psi : [0,1) \to (0,\infty)$ *be an increasing function such that*

$$\int_0^1 \log^+ \log^+ \psi(t) dt < \infty. \tag{1}$$

If $f(z) = \sum a_n z^n$ *and* $|f(z)| \leq \psi(|z|)$ *on* $D(w,r) \cap \mathbb{D}$ *for some* $w \in \mathbb{T}$ *and* $r > 0$*, then* $f \notin \mathcal{U}$*.*

The Main Result

Theorem 1

Let $\psi : [0,1) \to (0,\infty)$ *be an increasing function such that*

$$\int_0^1 \log^+ \log^+ \psi(t)dt < \infty. \tag{1}$$

If $f(z) = \sum a_n z^n$ *and* $|f(z)| \le \psi(|z|)$ *on* $D(w,r) \cap \mathbb{D}$ *for some* $w \in \mathbb{T}$ *and* $r > 0$*, then* $f \notin \mathcal{U}$.

Theorem 1

Theorem 1

The special case of Theorem 1 where the inequality $|f(z)| \leq \psi(|z|)$ is required to hold on all of $\mathbb{D}$ is due to Melas (2000), who also showed that condition (1) is close to being sharp.

Consequences: Picard's Property of Universal Series

Consequences: Picard's Property of Universal Series

Corollary 2

Let $f \in \mathcal{U}$. Then, for every $w \in \mathbb{T}$ and $r > 0$, the function f assumes every complex value, with at most one exception, infinitely often on $D(w, r) \cap \mathbb{D}$.

Consequences: Picard's Property of Universal Series

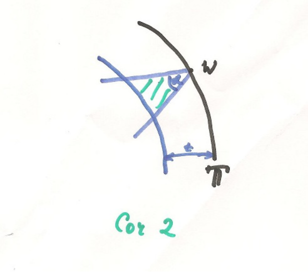
Corollary 2

Let $f \in \mathcal{U}$. Then, for every $w \in \mathbb{T}$ and $r > 0$, the function f assumes every complex value, with at most one exception, infinitely often on $D(w, r) \cap \mathbb{D}$.

Costakis and Melas had previously proved that f assumes every complex value, with at most one exception, infinitely often on $\mathbb{D}$; their argument shows that there is at least one point $w \in \mathbb{T}$ with the stated Picard-type property.

Costakis and Melas had previously proved that f assumes every complex value, with at most one exception, infinitely often on $\mathbb{D}$; their argument shows that there is at least one point $w \in \mathbb{T}$ with the stated Picard-type property. The exceptional value can actually arise: it was shown by Costakis and Melas (2000) that there exist zero-free members of $\mathcal{U}$.

Costakis and Melas had previously proved that f assumes every complex value, with at most one exception, infinitely often on $\mathbb{D}$; their argument shows that there is at least one point $w \in \mathbb{T}$ with the stated Picard-type property. The exceptional value can actually arise: it was shown by Costakis and Melas (2000) that there exist zero-free members of $\mathcal{U}$. Further, any function f in $\mathcal{U}$ must assume all but one complex value in any angle at "most" (set of the 2d category) boundary points .



Picard's Property: Quantitative Version

Picard's Property: Quantitative Version

Remark 1

We can give a quantitative version of Corollary 2, which improves Melas' (2000) result as follows. Let $f \in \mathcal{U}$*. Then, for any* $w \in \mathbb{T}$, $r > 0$, $\kappa \geq 1$, *and all but at most one complex number* a*, the distinct zeros* $(z_j(a))$ *of* $f - a$ *in* $D(w, r) \cap \mathbb{D}$ *satisfy*

$$\sum (1 - |z_j(a)|)^{\kappa} = \infty. \tag{2}$$

Picard's Property: Quantitative Version

Remark 1

We can give a quantitative version of Corollary 2, which improves Melas' (2000) result as follows. Let $f \in \mathcal{U}$. *Then, for any* $w \in \mathbb{T}$, $r > 0$, $\kappa \geq 1$, *and all but at most one complex number* a, *the distinct zeros* $(z_j(a))$ *of* $f - a$ *in* $D(w, r) \cap \mathbb{D}$ *satisfy*

$$\sum (1 - |z_j(a)|)^{\kappa} = \infty. \tag{2}$$

To prove this, suppose that the above series converges for two distinct choices of a.

Picard's Property: Quantitative Version

Remark 1

We can give a quantitative version of Corollary 2, which improves Melas' (2000) result as follows. Let $f \in \mathcal{U}$*. Then, for any* $w \in \mathbb{T}$, $r > 0$, $\kappa \geq 1$, *and all but at most one complex number a, the distinct zeros* $(z_j(a))$ *of* $f - a$ *in* $D(w, r) \cap \mathbb{D}$ *satisfy*

$$\sum (1 - |z_j(a)|)^{\kappa} = \infty. \tag{2}$$

To prove this, suppose that the above series converges for two distinct choices of a. Then $\log |f(z)| \leq C(1 - |z|)^{-\kappa-1}$ on $D(w, r/2) \cap \mathbb{D}$. (This relies on Nevanlinna value distribution theory, combined with a suitable conformal mapping from $D(w, r) \cap \mathbb{D}$ to $\mathbb{D}$.)

Picard's Property: Quantitative Version

Remark 1

We can give a quantitative version of Corollary 2, which improves Melas' (2000) result as follows. Let $f \in \mathcal{U}$*. Then, for any* $w \in \mathbb{T}$, $r > 0$, $\kappa \geq 1$, *and all but at most one complex number* a*, the distinct zeros* $(z_j(a))$ *of* $f - a$ *in* $D(w, r) \cap \mathbb{D}$ *satisfy*

$$\sum (1 - |z_j(a)|)^{\kappa} = \infty. \qquad (2)$$

To prove this, suppose that the above series converges for two distinct choices of a. Then $\log |f(z)| \leq C(1 - |z|)^{-\kappa - 1}$ on $D(w, r/2) \cap \mathbb{D}$. (This relies on Nevanlinna value distribution theory, combined with a suitable conformal mapping from $D(w, r) \cap \mathbb{D}$ to $\mathbb{D}$.) Theorem 1 can now be invoked to obtain a contradiction.

Consequences: Growth of Universal Series

Consequences: Growth of Universal Series

Membership of $\mathcal{U}$ is incompatible with any local Bergman-type integrability condition.

Consequences: Growth of Universal Series

Membership of $\mathcal{U}$ is incompatible with any local Bergman-type integrability condition.

Corollary 3

Let $f \in \mathcal{U}$. Then, for every $w \in \mathbb{T}$ and $r > 0$, and every $\beta > -1$,

$$\int_{D(w,r)\cap\mathbb{D}} \log^+ |f(z)| \left(1 - |z|^2\right)^{\beta} dA(z) = \infty.$$

In particular, f does not belong to any Bergman or Bergman-Nevanlinna class on $\mathbb{D}$.

Consequences: Growth of Universal Series

Membership of $\mathcal{U}$ is incompatible with any local Bergman-type integrability condition.

Corollary 3

Let $f \in \mathcal{U}$. Then, for every $w \in \mathbb{T}$ and $r > 0$, and every $\beta > -1$,

$$\int_{D(w,r)\cap\mathbb{D}} \log^+ |f(z)| \left(1 - |z|^2\right)^\beta dA(z) = \infty.$$

In particular, f does not belong to any Bergman or Bergman-Nevanlinna class on $\mathbb{D}$.

Subharmonicity of $\log^+ |f|$ yields that

$$\log^+ |f(\zeta)| \leq \frac{4}{\pi(1-|\zeta|)^2} \int_{D(\zeta,(1-|\zeta|)/2)} \log^+ |f(z)| \, dA(z) \leq \frac{C(\beta)}{(1-|\zeta|)^{\beta+2}}$$

$C(\beta)$ is a positive constant depending only on β.

Consequences: Growth of Universal Series

Membership of $\mathcal{U}$ is incompatible with any local Bergman-type integrability condition.

Corollary 3

Let $f \in \mathcal{U}$. Then, for every $w \in \mathbb{T}$ and $r > 0$, and every $\beta > -1$,

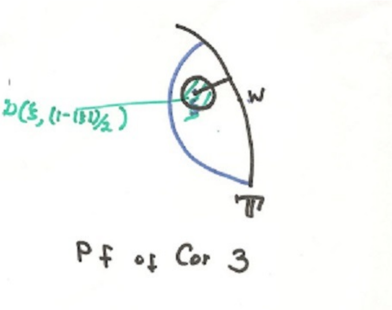
$$\int_{D(w,r)\cap\mathbb{D}} \log^+ |f(z)| \left(1 - |z|^2\right)^\beta dA(z) = \infty.$$

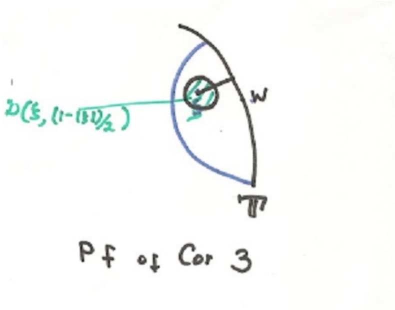
In particular, f does not belong to any Bergman or Bergman-Nevanlinna class on $\mathbb{D}$.

Subharmonicity of $\log^+ |f|$ yields that

$$\log^+ |f(\zeta)| \le \frac{4}{\pi(1-|\zeta|)^2} \int_{D(\zeta,(1-|\zeta|)/2)} \log^+ |f(z)|\, dA(z) \le \frac{C(\beta)}{(1-|\zeta|)^{\beta+2}}$$

$C(\beta)$ is a positive constant depending only on β. It now follows again from Theorem 1 that $f \notin \mathcal{U}$.

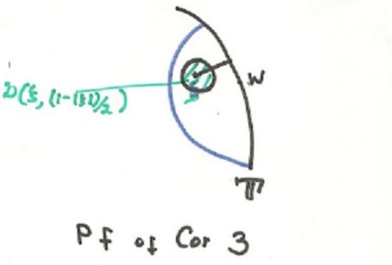

D(ξ, (1-|ξ|)/2)
w
T
Pf of Cor 3



Subharmonicity of $\log^+|f|$ yields that

$$\log^+|f(\zeta)| \leq \frac{4}{\pi(1-|\zeta|)^2}\int_{D(\zeta,(1-|\zeta|)/2)} \log^+|f(z)|\,dA(z) \leq \frac{C(\beta)}{(1-|\zeta|)^{\beta+2}}$$

$C(\beta)$ is a positive constant depending only on β.



Subharmonicity of $\log^+|f|$ yields that

$$\log^+|f(\zeta)| \leq \frac{4}{\pi(1-|\zeta|)^2}\int_{D(\zeta,(1-|\zeta|)/2)} \log^+|f(z)|\,dA(z) \leq \frac{C(\beta)}{(1-|\zeta|)^{\beta+2}}$$

$C(\beta)$ is a positive constant depending only on β. It now follows again from Theorem 1 that $f \notin \mathcal{U}$.

Beurling - Domar - Levinson - Sjoberg Theorem, 1939 -1952

Beurling - Domar - Levinson - Sjoberg Theorem, 1939 -1952

Theorem 4

Let $\psi : [0,1) \to (0,\infty)$ *be an increasing function such that*

$$\int_0^1 \log^+ \log^+ \psi(t)dt < \infty. \qquad (3)$$

If $\mathfrak{F} := \{f(z)$ *analytic in* $\mathbb{D}$, *such that* $|f(z)| \leq \psi(|z|)\}$ *on* $\mathbb{D}$, *then* $\mathfrak{F}$ *is a normal family.*

Beurling - Domar - Levinson - Sjoberg Theorem, 1939 -1952

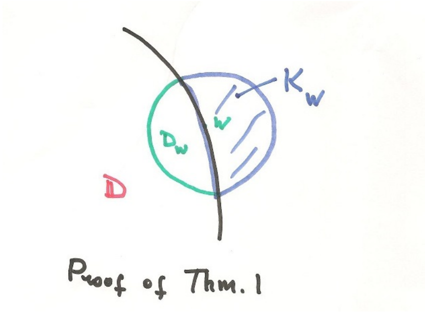
Theorem 4

Let $\psi : [0,1) \to (0,\infty)$ *be an increasing function such that*

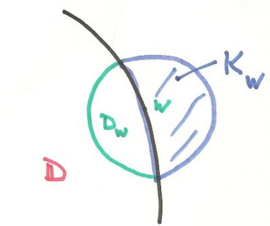
$$\int_0^1 \log^+ \log^+ \psi(t)dt < \infty. \tag{3}$$

If $\mathfrak{F} := \{f(z)$ *analytic in* $\mathbb{D}$*, such that* $|f(z)| \leq \psi(|z|)\}$ *on* $\mathbb{D}$*, then* $\mathfrak{F}$ *is a normal family.*

Theorem 1



To apply Theorem 4 for the proof of Theorem 1, we show that the partial Taylor sums $S_N(z)$ of f are controlled as follows : $|S_N(z)| \leq C_{w,r}\psi(|z|)$ in $D(w,r) \cap \mathbb{D}$.



To apply Theorem 4 for the proof of Theorem 1, we show that the partial Taylor sums $S_N(z)$ of f are controlled as follows : $|S_N(z)| \leq C_{w,r}\psi(|z|)$ in $D(w,r) \cap \mathbb{D}$. Then, by universality of f, we can choose a subsequence S_{N_k} converging to 0 uniformly outside of $\mathbb{D}$, and apply Theorem 4 to it, concluding that $f \equiv 0$, a contradiction.

Universal Homogeneous Harmonic Series

Universal Homogeneous Harmonic Series

Theorem 5

Let $\psi : [0,1) \to (0,\infty)$ *be an increasing function such that*

$$\int_0^1 \log^+ \psi(t)dt < \infty. \tag{4}$$

If $h(x)$ *is a harmonic function in the unit ball* $\mathbb{B}(0,1)$ *in* $\mathbb{R}^d$ *and* $|h(x)| \leq \psi(|x|)$ *on* $B(w,r) \cap \mathbb{B}$ *for some* $w \in \mathbb{S}^{d-1}$ *and* $r > 0$*, then* $f \notin \mathcal{U}_H$.

Universal Homogeneous Harmonic Series

Theorem 5

Let $\psi : [0,1) \to (0,\infty)$ *be an increasing function such that*

$$\int_0^1 \log^+ \psi(t)dt < \infty. \tag{4}$$

If $h(x)$ *is a harmonic function in the unit ball* $\mathbb{B}(0,1)$ *in* $\mathbb{R}^d$ *and* $|h(x)| \leq \psi(|x|)$ *on* $B(w,r) \cap \mathbb{B}$ *for some* $w \in \mathbb{S}^{d-1}$ *and* $r > 0$*, then* $f \notin \mathcal{U}_H$.

The reason why only one "log" appears in (4), in contrast to (3), is that we apply Domar's result to subharmonic functions of the form $|h|$ rather than $\log |f|$.

THANK YOU!

THANK YOU!

Happy Birthday, Alex!