Boundary Behavior of Universal Taylor Series
Perspectives of Modern Complex Analysis
Bedlewo, Poland, July 2014

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Outline

1. Introduction: Universality, Examples

Boundary Behavior = “Misbehavior” of Universal Series

A Proof of the Main Result

Universal Polynomial Expansions of Harmonic Functions
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3. A Proof of the Main Result
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2. Boundary Behavior = “Misbehavior” of Universal Series
3. A Proof of the Main Result
4. Universal Polynomial Expansions of Harmonic Functions
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- M. Fekete (1886 - 1957), in a paper written by J. Pal in 1914 constructed $U := \sum_{1}^{\infty} a_n x^n$, $a_n \in \mathbb{R}$, ROC = 0 such that $\forall g \in C_{\mathbb{R}}[-1, 1], g(0) = 0, \exists$ a subsequence of partial sums $s_{n_k}$ of $U$ s.t. $s_{n_k} \Rightarrow g$ on $[-1, 1]$. 
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- George D. Birkhoff (1884 - 1944) showed (1929) existence of an entire function \( f(z) \) whose translates \( f(z + n), \ n \in \mathbb{N} \) can approximate any entire function uniformly on compact subsets of \( \mathbb{C} \).
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A. I. Seleznev (1952) constructed a power series \( \sum a_n z^n \), with \( \text{ROC} = 0 \): \( \forall \) compact \( K \subset \mathbb{C} \setminus \{0\} : \mathbb{C} \setminus K \) is connected, \( \forall g \in A(K) \exists s_{n_k} := \sum_{1}^{n_k} a_n z^n \), a subsequence of partial sums, \( s_{n_k} \Rightarrow g \) on \( K \).
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V. Nestorides (1990) proved that universality holds on \( \{|z| \geq r\} \). He also showed that the set of universal power series \( \mathcal{U} \) is a dense \( G_\delta \) subset of the space of all holomorphic functions on the disk endowed with the topology of uniform convergence on compact subsets.
Place of Action

open disk
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open disk

or

Universality
The Main Result

Theorem 1

Let $\psi: [0, 1) \to (0, \infty)$ be an increasing function such that

$$\int_0^1 \log \psi(t) \, dt < \infty. \quad (1)$$

If $f(z) = \sum a_n z^n$ and $|f(z)| \leq \psi(|z|)$ on $D(w, r) \cap D$ for some $w \in T$ and $r > 0$, then $f/\psi \in U$. 


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Introduction: Universality, Examples  Boundary Behavior = “Misbehavior” of Universal Series  A Proof of the Main Result  Universal Polynomial Expansions of Harmonic Functions

The special case of Theorem 1 where the inequality $|f(z)| \leq \psi(|z|)$ is required to hold on all of $D$ is due to Melas (2000), who also showed that condition (1) is close to being sharp.
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Consequences: Picard’s Property of Universal Series

Let $f \in U$. Then, for every $w \in T$ and $r > 0$, the function $f$ assumes every complex value, with at most one exception, infinitely often on $D(w, r) \cap D$. 

**Corollary 2**
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*Theorem 1*
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Picard’s Property: Quantitative Version

Remark 1

We can give a quantitative version of Corollary 2, which improves Melas’ (2000) result as follows. Let $f \in U$. Then, for any $w \in \mathbb{T}$, $r > 0$, and $\kappa \geq 1$, and all but at most one complex number $a$, the distinct zeros $z_j(a)$ of $f - a$ in $D(w, r) \cap D$ satisfy

$$\sum (1 - |z_j(a)|) \kappa = \infty.$$ (2)

To prove this, suppose that the above series converges for two distinct choices of $a$. Then $\log |f(z)| \leq C (1 - |z|)^{\kappa - 1}$ on $D(w, r/2) \cap D$. (This relies on Nevanlinna value distribution theory, combined with a suitable conformal mapping from $D(w, r) \cap D$ to $D$.) Theorem 1 can now be invoked to obtain a contradiction.
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Consequences: Growth of Universal Series

Let $f \in U$. Then, for every $w \in T$ and $r > 0$, and every $\beta > -1$,
$$\int_{D(w, r)} \log \left| f(z) \right| \left(1 - |z|^2\right)^{\beta} dA(z) = \infty.$$  
In particular, $f$ does not belong to any Bergman or Bergman-Nevanlinna class on $D$.

Subharmonicity of $\log \left| f(\zeta) \right|$ yields that
$$\log \left| f(z) \right| \leq 4\pi \left(1 - |\zeta|^2\right)^2 \int_{D(\zeta, (1 - |\zeta|^2)/2)} \log \left| f(z) \right| dA(z) \leq C(\beta) \left(1 - |\zeta|^2\right)^{\beta + 2}.$$  
$C(\beta)$ is a positive constant depending only on $\beta$. It now follows again from Theorem 1 that $f \not\in U$. 

Consequences: Growth of Universal Series

Membership of $U$ is incompatible with any local Bergman-type integrability condition.
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**Corollary 3**

Let $f \in \mathcal{U}$. Then, for every $w \in \mathbb{T}$ and $r > 0$, and every $\beta > -1$,

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Pf of Cor 3

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\[ \pi \]

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Beurling - Domar - Levinson - Sjoberg Theorem, 1939-1952

Theorem 4

Let $\psi: [0, 1) \to (0, \infty)$ be an increasing function such that

$$\int_0^1 \log_+ \log_+ \psi(t) \, dt < \infty.$$  \hfill (3)

If $F := \{f(z) \text{ analytic in } D, \text{ such that } |f(z)| \leq \psi(|z|)\}$ on $D$, then $F$ is a normal family.
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If $\mathcal{F} := \{ f(z) \text{ analytic in } \mathbb{D}, \text{ such that } |f(z)| \leq \psi(|z|) \}$ on $\mathbb{D}$, then $\mathcal{F}$ is a normal family.
To apply Theorem 4 for the proof of Theorem 1, we show that the partial Taylor sums $S_N(z)$ of $f$ are controlled as follows:

$$|S_N(z)| \leq C_{w,r} \psi(|z|) \text{ in } D(w, r) \cap \mathbb{D}.$$
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$$|S_N(z)| \leq C_{w,r} \psi(|z|) \text{ in } D(w, r) \cap \mathbb{D}.$$ 

Then, by universality of $f$, we can choose a subsequence $S_{N_k}$ converging to 0 uniformly outside of $\mathbb{D}$, and apply Theorem 4 to it, concluding that $f \equiv 0$, a contradiction.
Universal Homogeneous Harmonic Series

Theorem 5

Let \( \psi : [0,1) \rightarrow (0,\infty) \) be an increasing function such that
\[
\int_0^1 \log \psi(t) \, dt < \infty.
\]
(4)

If \( h(x) \) is a harmonic function in the unit ball \( B(0,1) \) in \( \mathbb{R}^d \) and
\[
|h(x)| \leq \psi(|x|)
\]
on \( B(w,r) \cap B \) for some \( w \in S^{d-1} \) and \( r > 0 \), then \( f \in UH \).

The reason why only one "log" appears in (4), in contrast to (3), is that we apply Domar's result to subharmonic functions of the form \( |h| \) rather than \( \log |f| \).
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THANK YOU!
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Happy Birthday, Alex!