Perspectives in modern complex analysis
Conference in honour of Alex Eremenko

Local properties of planar harmonic maps

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Planar harmonic maps

A map $\mathbb{R}^2 \supset W \xrightarrow{f} \mathbb{R}^2$ is called a **planar harmonic map** if both components of f are harmonic functions.

When W is simply connected, identifying $\mathbb{R}^2$ with $\mathbb{C}$, the map f can also be written in the complex form $p(z) + \overline{q(z)}$ where p and q are holomorphic functions on W.
Note that f is real analytic.

Here we are interested on local properties of germs $f : (\mathbb{C}, 0) \to (\mathbb{C}, 0)$ of planar harmonic maps, especially:

1. Topological properties (inspired by works of Lyzzaik).
2. Links between numerical analytic invariants.

Notations

The critical set C_f which is the vanishing locus of the Jacobian J_f.

The critical value set $\mathbf{V_f} = f(C_f)$. It will play an important role.

Assumptions

1. $f^{-1}(0) = \{0\}$ (This condition is equivalent to light in the sense of Lyzzaik).

2. C_f is a smooth curve at the origin.

Condition 2. implies that C_f is not a single point. Otherwise, one can prove that up to C^1 change of coordinates, f is holomorphic or anti-holomorphic

Two examples

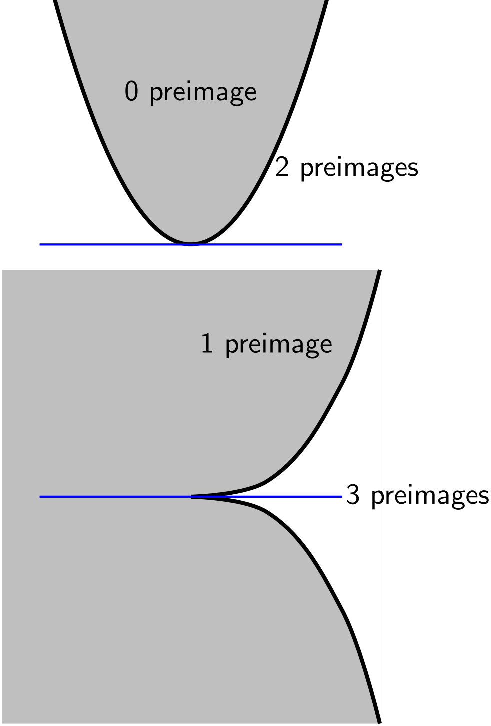
A. $f(z) = \frac{1}{2}\big(z + \overline{z} + i(z^2 + \overline{z}^2)\big)$
in real coordinates: $f(x, y) = (x, x^2 - y^2)$.

$C_f = \mathbb{R}$, $V_f = \left\{(x, y) | y = x^2\right\}$

B. $f(z) = -i\big(z - \overline{z} + iz^2 - \frac{2}{3}z^3\big)$

$C_f = \mathbb{R}$, $V_f = \left\{(t^2, \frac{2}{3}t^3), t \in \mathbb{R}\right\}$.

The first example is a **fold** and the second a **cusp**.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The topological model

We generalize the previous remarks in order to give a geometric model.

We need:

1. The local topological degree $\mathbf{d}$ of f. It is the winding number for a small circle around the origin.
It can be proved that in our situation, $\mathbf{d} = 0, 1, -1$.

2. $\mathbf{m}$ which is by definition the minimal degree of a monomial appearing in the power expansion of f.

In our examples, $m = 1$.
For the fold, $d = 0$ and for the cusp $d = 1$.

The topological model

$z^m + O(z^{m+1}) - \overline{z}^m$	$m + d$ odd	$m + d$ even
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V_f	convex	cusp
$\#\{\text{fibers}\}$	$m+1, \; m-1$	$m+2, \; m$
$f \sim \begin{matrix} z^{2n^+-1}\|_{y>0} \\ \bar{z}^{2n^--1}\|_{y<0} \end{matrix}$	$\begin{cases} n^+ - n^- = \|d\| \\ n^+ + n^- = m+1 \end{cases}$	$\begin{cases} n^+ - n^- = \|d\| \\ n^+ + n^- = m+2 \end{cases}$

Topological classification

1. Fix m.

First case : $d+m$ odd, then
V_f convex.
cardinality of the fibers: $m+1$ and $m-1$.

Second case: $d+m$ even, then
V_f cusp of the first kind.
cardinality of the fibers: $m+2$ and m.
The fibers of maximal cardinality come from the white region (due to the harmonicity).

2. The map f is topologically equivalent to:

$(r,\theta) \to re^{i(2n^+-1)\theta}$ for $0 \leq \theta \leq \pi$
$(r,\theta) \to re^{i(2n^--1)\theta}$ for $\pi \leq \theta \leq 2\pi$
where n^+ and n^- satisfy:
$n^+ - n^- = |d|$.
and $n^+ + n^-$ is the maximal cardinality for the fiber.

Numerical invariant j

Let $\gamma(t)$ a regular analytic parametrization of C_f.

By definition, j is defined as the order of $\beta(t) = f \circ \gamma(t)$.

Theorem

In a frame where the x-axis is tangent to V_f at the origin, the curve $\beta(t)$ takes the form:

$$\left(At^j + h.o.t, \quad Bt^{j+1} + h.o.t\right)$$

where $A, B \neq 0$.

We say that V_f has order pair $(j, j+1)$. Such an order pair is a real analytic invariant of f.

Thus:

1. j even $\implies V_f$ cusp.
   j odd $\implies V_f$ convex.
2. d and $j + m$ have same parity.

Complexification and the invariant μ

The **complexification** of a light harmonic germ is also light.

The **local multiplicity** μ is defined (for the complexification) as the cardinality of a generic fiber. The following relation between μ, j, m is somewhat surprising.

Theorem

$$\mu = j + m^2$$

The proof uses the theory of Milnor fibration.

Remark on $(j, j+1)$.

The order pair can be viewed in the complex setting as a Puiseux pair. It characterizes topologically the complexification of V_f.

What about real analytic planar maps ?

An easy observation is that in our situation, the case $m = 1$ corresponds to the fact that the gradient of the Jacobian doesn't vanish at the origin. We call critically regular a germ F of planar

(real or complex) analytic map defined at the origin satisfying the previous nonvanishing property.

In this case, we have (using the previous notations):

Theorem

1. $\mu = j + 1$.
2. F has an order pair (Puiseux pair) $(j, j+1)$.
3. In the real case, F is topologically a fold or a cusp.

Computation of j

Inspired by works of Whitney, we give a way to compute j (and then also $|d|$) in the case of a critically regular planar analytic germ F. It leads to an algorithm in the polynomial case.

Notations For a planar real analytic germ $f : (\mathbb{C}, 0) \to (\mathbb{C}, 0)$ define $\nabla f = (f_z, f_{\overline{z}})$.
Denote by J the Jacobian of F.

We set recursively: (det is the determinant)

$$L_1 = \det(\nabla f, \nabla J),\ L_2 = \det(\nabla L_1, \nabla J), \cdots, L_k = \det(\nabla L_{k-1}, \nabla J) \cdots$$

Theorem
The invariant j is the first integer n for which $L_n(0) \neq 0$.

Class of examples

1. m is a strictly positive integer.

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2. m and k are strictly positive integers.

$$f(z) = z^m - \overline{z}^m + i\frac{m}{m+1}(z^{m+1} - \overline{z}^{m+1}) + i\frac{m}{m+k}(z^{m+k} + \overline{z}^{m+k})$$
$$-\frac{m}{m+k+1}(z^{m+k+1} + \overline{z}^{m+k+1})$$

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In this case $j = k + m$ and $|d|$ is given by the parity of k.

THANKS FOR YOUR ATTENTION

Dziękuję