Several questions on normal families of harmonic functions

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Toy question

Assume f is a holomorhpic function in $P = \{z \in \mathbb{C} : |Im(z)| \leq 1\}$ such that

$$|f(x+iy)| \leq \frac{1}{|y|}$$

for $z = x + iy \in P$. **Why $|f(0)| \leq 10$?**

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Answer

Because $\log|f|$ is subharmonic.

$$\log|f|(0) \leq \frac{1}{2\pi}\int_{|z|=1} \log|f| \leq \frac{1}{2\pi}\int_{|z|=1} \log\frac{1}{|y|}|dz| \leq \log 10.$$

The Levinson log log theorem.

Let P be a rectangle $(-a,a)\times(-b,b)$ in $\mathbb{R}^2 \simeq \mathbb{C}$ and let $M:(0,b)\to[e,+\infty)$ be a decreasing function. Consider the set $\mathcal{F}_M$ of all functions f holomorphic in P such that $|f(x,y)|\le M(|y|)$, $(x,y)\in P$.

Theorem (The Levinson log log theorem)

If $\int\limits_0^b \log\log M(y)dy < +\infty$, then $\mathcal{F}_M$ is a normal family in P (i.e. is uniformly bounded on compact subsets of P).

Examples: $M(y) = 1/|y|$, $\exp(1/|y|)$.

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Sharpness(Levinson, ... , Beurling)

For regular (continuous and decreasing) majorants M the family $\mathfrak{F}_M$ is normal if and only if $\int_0^b \log\log M(y)dy < +\infty$.

Let P denote the set $\{(x, y) : x \in \mathbb{R}^{n-1}, y \in \mathbb{R}, |x| < R, |y| < H\}$, where R and H are some positive numbers.

Theorem ("Harmonic" analogue of the Levinson log log theorem.)

Assume a function M: $(0, H) \to \mathbb{R}_+$ *is decreasing and*

$$\int_0^H \log^+ \log^+ M(y) dy < +\infty. \tag{1}$$

Then the set $\mathcal{H}_M$ *of all functions* u *harmonic in* P *and satisfying* $|u(x, y)| \leq M(|y|)$, $(x, y) \in P$, *is uniformly bounded on any compact subset of* P.

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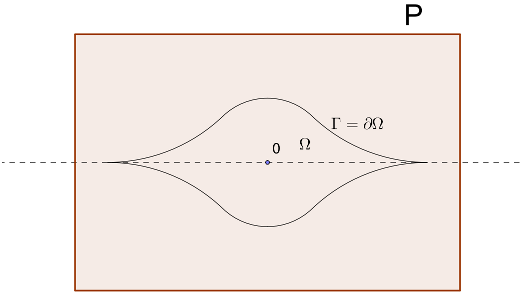
The main obstacle in higher dimensions

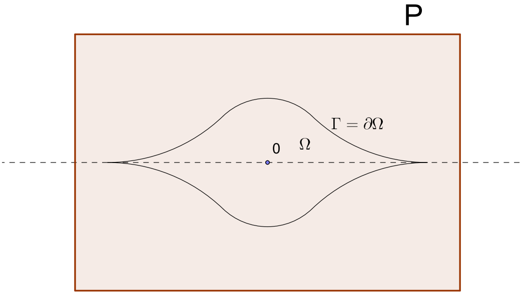
$\log|\nabla u|$ is not necessarily subharmonic if $n \geq 3$.

Idea of the proof of the "holomorphic" Levinson theorem via harmonic measure estimates.

Let $d\omega$ denote the harmonic measure of Ω viewed from 0.



$$\log|f|(0) \leq \int_\Gamma \log|f|(z)d\omega(z) \leq \int_\Gamma \log M(|y|)d\omega(z).$$



$$\log|f|(0) \leq \int_\Gamma \log|f|(z)d\omega(z) \leq \int_\Gamma \log M(|y|)d\omega(z).$$

Idea: to choose Γ so that $\int_\Gamma \log M(|y|)d\omega(z) < +\infty$
Tool: harmonic measure estimates (Ahlfors' distortion theorem, description of radial projections of harmonic measures of star-shaped domains (due to Rashkovskii)).

The estimates in the Levinson loglog theorem.

$\mathcal{F}_M = \{f \in Hol(P) : |f(x,y)| \le M(y), (x,y) \in P\}$.
$M : (-b, b) \to [e, +\infty)$.

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Domar's estimate.

Let $\widetilde{M} := \log M$. $F(t) := \lambda_1(\{y \in (-b,b) : \widetilde{M}(y) \ge t\})$.

$$\sum_{i=0}^{+\infty} F(2^i) < +\infty \text{ iff } \int_{-b}^{b} \log\log M(y)dy < +\infty.$$

If $z \in P$ and $\sum\limits_{i=-1}^{+\infty} F(2^i C) < \frac{\pi}{8} dist(z, \partial P)$ for a positive constant C. Then $|f|(z) \le exp(C)$.

"Harmonic" analogue of the Levinson log log theorem.

Theorem (Dyn'kin,1995)

A strictly positive decreasing function M on $(0,1)$, with $M(+0) = +\infty$ is called a regular majorant if

1. $\log M(e^{-t})$ *is a convex function of* $t > 0$.
2. $M(r)r \geq M(qr)$, $0 < r \leq 1$, *for some constant* q.

Assume a function $M\colon (0,1) \to \mathbb{R}_+$ *is a regular majorant and*

$$\int_0^1 \log^+ \log^+ M(y)dy < +\infty.$$

Then the set $\mathcal{H}_M$ *of all functions* u *harmonic in* $P = \{(x,y) : x \in \mathbb{R}^{n-1}, y \in \mathbb{R}, |x| < 1, |y| < 1\}$ *and satisfying* $|u(x,y)| \leq M(|y|)$, $(x,y) \in \Omega$, *is a normal family.*

Let P denote the set $\{(x, y) : x \in \mathbb{R}^{n-1}, y \in \mathbb{R}, |x| < R, |y| < H\}$, where R and H are some positive numbers.

Theorem (A.L., 2014)

Assume a function M: $(0, H) \to \mathbb{R}_+$ is decreasing and

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Then the set $\mathcal{H}_M$ of all functions u harmonic in P and satisfying $|u(x, y)| \le M(|y|)$, $(x, y) \in P$, is uniformly bounded on any compact subset of P.

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The main obstacle in higher dimensions

$\log |\nabla u|$ is not necessarily subharmonic if $n \geq 3$.

If $n = 2$

- If $z = (x, y) \in P$ and $B_{y/2}(z) \subset P$, then $\sup\limits_{B_{y/2}(z)} |u| \leq M(y/2)$.

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- $f := u_x - iu_y$ is holomorphic and $|f|(z) = |\nabla u|(z) \leq \widetilde{M}(y)$.
- $\int \log^+ \log^+ \widetilde{M} < +\infty \iff \int \log^+ \log^+ M < +\infty$

Let P_ε denote $\{(x,y) : x \in \mathbb{R}^{n-1}, y \in \mathbb{R}, |x| < \varepsilon, |y| < H\}$. Let $\rho := |x|$ and $h := y$. A function u is called axially-symmetric if $u = u(\rho, h)$.

Lemma

Assume $u = u(\rho, h)$ is an axially-symmetric harmonic function and $|u(x,y)| \le M(|y|)$ in the truncated cylinder P_ε, where M is decreasing and $\int\limits_0^H \log^+ \log^+ M(y)dy < +\infty$. Then there is a constant $C = C(M, H, \varepsilon)$ such that $|u(0,y)| < C$ for any $y \in (-H+\varepsilon, H-\varepsilon)$.

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Euler-Darboux equation

$$u_{\rho\rho} + u_{hh} + \frac{n-2}{\rho} u_\rho = 0$$

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- Apply 2-dimensional Levinson $\log\log$ theorem...
- Then for any compact set K $\exists C = C(K, M) : |\nabla v| \leq C$. Since $v_\rho(0, h) = u(0, h)$ the lemma is proved.

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- Apply 3-dimensional case, which follows from 4-dimensional case.
- The case of even n follows immediately.

Question on one-sided estimates.

Assume that z_0 is a point in the square $Q = (-1,1) \times (-1,1)$ and M is a positive and decreasing (regular) function on $(0,1)$. Under what assumptions on M the family F_M^+ of all functions f holomorphic in Q, and satisfying $Im(f(z)) \leq M(|Im(z)|)$, $f(z_0) = 0$ is normal?

If u is a harmonic function in $\mathbb{R}^n$, then $\log|\nabla u|$ is not necessarily subharmonic if $n \geq 3$.

Remark 1

Let B denote the unit ball centered at 0. Let C be a relatively open subset of ∂B. Suppose $|\nabla u|$ is less than ε on C and $|\nabla u| \leq 1$ on ∂B. Then there exists $\alpha = \alpha(C) \in (0,1]$: $|\nabla u|(0) \leq \varepsilon^{\alpha}$.

Remark 2

Suppose u is a harmonic function in $\mathbb{R}^n_+$ and $F = \nabla u$ satifies $|F| \leq 1$ on $\partial\mathbb{R}^n_+$. If $|F|(x) \leq exp(|x|^{1-\varepsilon})$ for some $\varepsilon \in (0,1)$ (or $|F|(x) \leq exp(o|x|)$), then $|F| \leq 1$ on $\mathbb{R}^n_+$.

Thank You!