On the topological differences between the Mandelbrot set and the tricorn

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Basic definitions

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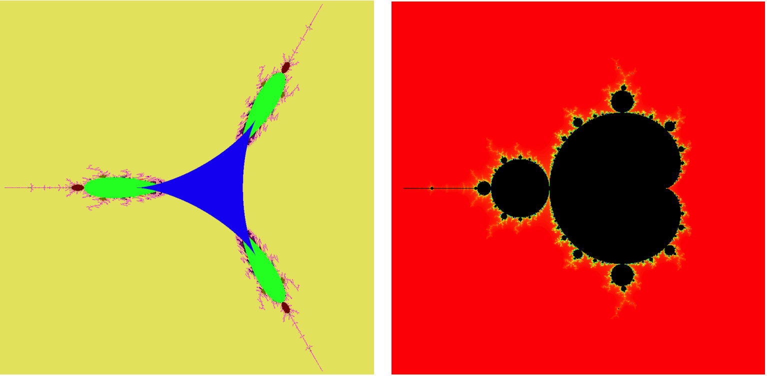
There are, however, striking differences between the topological features of the tricorn and those of the Mandelbrot set.

The tricorn and the Mandelbrot set



Left: The tricorn. Right: The Mandelbrot set.

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Theorem (Nakane)

The map $\Phi : \mathbb{C} \setminus \mathcal{T} \to \mathbb{C} \setminus \overline{\mathbb{D}}$, defined by $c \mapsto \phi_c(c)$ (where ϕ_c is the Böttcher coordinate near ∞) is a real-analytic diffeomorphism.

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The previous theorem allows us to define parameter rays of the tricorn as pre-images of radial lines in $\mathbb{C} \setminus \overline{\mathbb{D}}$ under the map Φ.

Anti-holomorphic Fatou Coordinate, Equator and Ecalle height

Lemma

Suppose z_0 *is a parabolic periodic point of odd period of* f_c *with only one petal and* U *is an immediate basin of attraction. Then there exists a petal* $V \subset U$ *and a univalent map* $\Phi \colon V \to \mathbb{C}$ *conjugating the first return map to* $\bar{z} + \frac{1}{2}$*. This map* Φ *is unique up to horizontal translation.*

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Non-Landing Parameter rays

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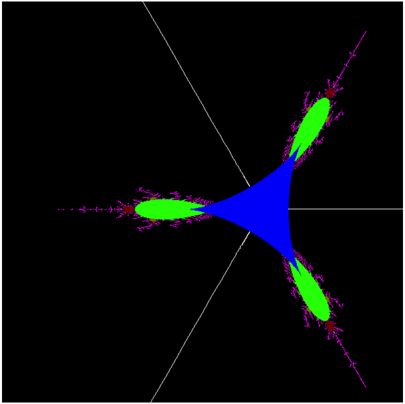

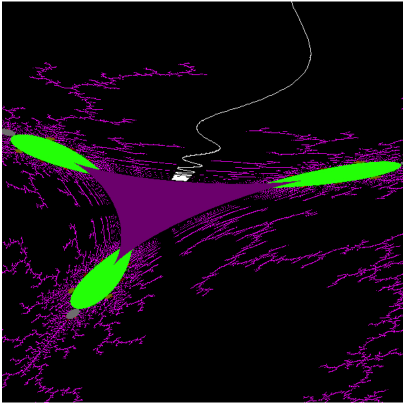

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- Landing of a parameter ray implies that the corresponding dynamical ray would project to a round circle in the repelling Ecalle cylinder.

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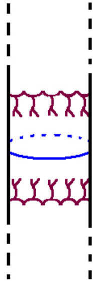

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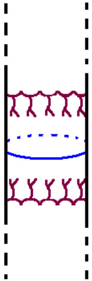

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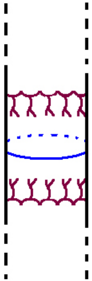

- This symmetry would force the rational lamination of the polynomial to be invariant under a 'suitable' affine transformation.
- An easy combinatorial argument now proves that this happens only on the boundary of the period 1 hyperbolic component.

Centers are not equidistributed w.r.t. the harmonic measure

Theorem (M)

For $d \geq 2$, every period 1 parabolic arc of $\mathcal{M}_d^*$ contains an undecorated sub-arc.

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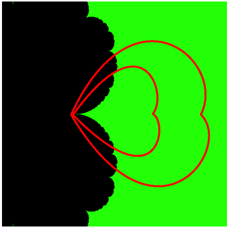

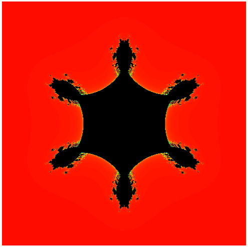

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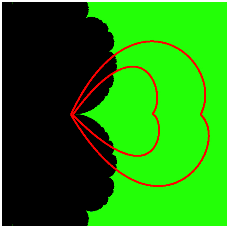

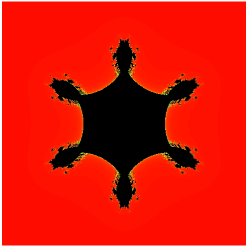

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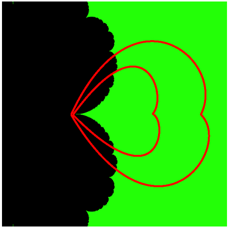

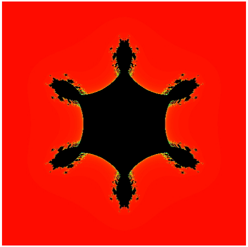

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- Transfer this round cylinder to the parameter plane (using parabolic perturbation techniques) to obtain an undecorated parabolic arc.

Thank you!