Differential Polynomials and Shared Values

Shahar Nevo
(Joint work with Jürgen Grahl)

Bedlewo, July 2014

Introduction

Bloch's Principle

Every condition which reduces $f \in M(\mathbb{C})$ to be a constant makes family of functions in $M(D)$ normal.

Let a, b, c be three distinct values in $\hat{\mathbb{C}}$.

Picard's Theorem

If $f \in M(\mathbb{C})$, $f \neq a, b, c$, then f is constant.

Montel's Theorem

If $f \neq a, b, c$ for every $f \in \mathcal{F} \subset M(D)$, then $\mathcal{F}$ is normal.

Definition

If $f, g \in M(D)$ *and nonconstant and if* $a \in \hat{\mathbb{C}}$, *then* f *and* g *share* a *CM if* $f(z) = a \Leftrightarrow g(z) = a$ *with the same multiplicity.*

Global Sharing-Values Version (Nevanlinna, 1926)

If $f, g \in M(\mathbb{C})$ *share* a_1, a_2, a_3, a_4, a_5, *then* $f = g$.

Local Sharing Values Version

If each $f, g \in \mathcal{F} \subset M(D)$ *share* a, b, c, *then* $\mathcal{F}$ *is normal.*

Involving Derivatives

Fundamental Global Sharing-Values Result (Mues-Steinmetz, 1979)

If $f \in M(\mathbb{C})$ and f and f' share $a_1, a_2, a_3 \in \mathbb{C}$, then $f = f'$.

Gundersen, 1980:

Two nonzero values are enough.

Fundamental Local Sharing-Values Result (Schwick, 1992)

If $\mathcal{F} \subset M(D)$ and every $f \in \mathcal{F}$ shares with f' a_1, a_2, a_3, then $\mathcal{F}$ is normal.

Pang and Zalcman, 2000:

Two values are enough.

From now on all functions are meromorphic in $\mathbb{C}$.

A General Problem

Suppose P is a differential polynomial, f, g are nonconstant in $M(\mathbb{C})$ and $P[f]$, $P[g]$ share one or several values. Under which assumptions can we deduce that $f \equiv g$ or that f and g are closely related?

Global Theorem (Hayman, Clunie, Mues, Bergweiler-Eremenko, Zalcman)

If $n \in \mathbb{N}$ and $f \in M(\mathbb{C})$, $f^n \cdot f' \neq 1$, then f is constant.

Local Theorem (Yang-Chang, Gu, Pang, Oshkin)

If $f^n f' \neq 1$ for every $f \in \mathcal{F} \subset M(D)$, then $\mathcal{F}$ is normal.

Theorem (Global Sharing-Values Version, Yang-Hua, 1997)

Let f and g be non-constant meromorphic functions in $\mathbb{C}$ and $n \geq 11$ be an integer. Assume that $f^n f'$ and $g^n g'$ share a non-zero value CM. Then $f = cg$ for some $c \in \mathbb{C}$ satisfying $c^{n+1} = 1$ or fg is constant and $f(z) = e^{az+b}$ for certain $a, b \in \mathbb{C}$. If f and g are entire, this also holds for $n \geq 7$.

Theorem (Hayman, 1959)

Let $a \neq 0$, $b \in \mathbb{C}$, $n \geq 5$. If $\psi_f := f^n + af'$ satisfies $\psi_f \neq b$, then f is constant. If f is entire it is true for $n \geq 3$ and for $n = 2$ if $b = 0$.

Theorem (Dueringer, 1982)

The same is true for $\psi_f := f^n + af^{(k)}$ if $n \geq k + 4$, and if f is entire then it is true for $n \geq 3$, independently of k.

Main Results

Theorem (Grahl-N1)

Let f and g be non-constant meromorphic functions in $\mathbb{C}$, $a, b \in \mathbb{C} \setminus \{0\}$ and let n and k be natural numbers satisfying $n \geq 5k + 17$. Assume that the functions

$$\psi_f := f^n + af^{(k)} \qquad \textit{and} \qquad \psi_g := g^n + ag^{(k)}$$

share the value b CM. Then

$$\frac{\psi_f - b}{\psi_g - b} = \frac{f^n}{g^n} = \frac{af^{(k)} - b}{ag^{(k)} - b}$$

or

$$\frac{\psi_f - b}{\psi_g - b} = \frac{f^n}{ag^{(k)} - b} = \frac{af^{(k)} - b}{g^n} \tag{1}$$

or $f = g$, $f^{(k)} = g^{(k)} \equiv \frac{b}{a}$.

In fact, we believe that the case (1) cannot occur at all, but we were not able to prove this.

If we restrict ourselves to entire functions, we can weaken the assumption on n a bit, and we can exclude the case (1), namely, we have the following theorem.

Theorem (Grahl-N2)

Let f and g be non-constant entire functions, $a, b \in \mathbb{C} \setminus \{0\}$ and let n and k be natural numbers satisfying $n \geq 11$ and $n \geq k+2$. Assume that the functions ψ_f and ψ_g share the value b CM. Then

$$\frac{\psi_f - b}{\psi_g - b} = \frac{f^n}{g^n} = \frac{af^{(k)} - b}{ag^{(k)} - b}$$

or $f = g$, $f^{(k)} = g^{(k)} \equiv \frac{b}{a}$.

Basics of Nevanlinna Theory

Let $r \geq 0$ and f in $M(\mathbb{C})$.

$$n(r,f) := \#(\text{poles of } f \text{ in } \overline{\Delta}(o,r), \text{ including multiplicities})$$

$$\sum_k \log \frac{r}{|b_k|} = \int_0^r \frac{n(t,f) - n(o,f)}{t} dt$$

($\{b_k\}$ are the poles in $\overline{\Delta}(o,r)$, each counted with regard to its multiplicity.)

$$N(r,f) := \sum_k \log \frac{r}{|b_k|} + n(o.f)\log r$$

$$m(r,f) := \frac{1}{2\pi}\int_0^{2\pi} \log^+ |f(re^{i\theta})| d\theta$$

$$T(r,f) := N(r,f) + m(r,f)$$

$\overline{N}(r, f)$ is the same as $N(r, f)$ but each b_k is counted once and is obtained by considering $\overline{n}(r, f)$ instead of $n(r, f)$.

For $a \in \mathbb{C}$, we denote $n(r, a) = n(r, f, a) := n(r, \frac{1}{f-a})$, and

$$N(r, a) := N\left(r, \frac{1}{f-a}\right), \quad m(r, a) := m\left(r, \frac{1}{f-a}\right),$$

$$T(r, a) := T\left(r, \frac{1}{f-a}\right).$$

Fundamental Inequalities

$$N\left(r, \sum_1^n f_k\right) \leq \sum_1^n N(r, f_k), \qquad N(r, f_1 \dots f_n) \leq \sum_1^n N(r, f_k)$$

$$m\left(r, \sum_1^n f_k\right) \leq \sum_1^n m(r, f_k) + \log n$$

$$m(r, f_1 \dots f_n) \leq \sum_1^p m(r, f_k)$$

$$T(r, \sum_1^n f_k) \leq \sum T(r, f_k) + \log n$$

$$T(r, f_1 \dots f_n) \leq \sum_1^n T(r, f_k)$$

First Fundamental Theorem

$$T(r, f) = T(r, a) + O(1).$$

Second Fundamental Theorem

Later in the talk I will introduce a direct application.

By $S(r, f)$ we denote an arbitrary term of the form $o(T(r, f))$ for $r \to \infty$, r outside some set of finite measure.

Lemma on the Logarithmic Derivative

For $f \in M(\mathbb{C})$ and $n \in \mathbb{N}$, there exists a constant M such that $m\left(r, \frac{f^{(n)}}{f}\right) \leq M(\log^+ r + \log^+ T(r, f) + 1)$ for $r \in \mathbb{C} \setminus E$, where E has finite Lebesgue measure. If f has finite order, then $E = \emptyset$.

In particular, $m\left(r, \frac{f^{(n)}}{f}\right) = S(r, f)$.

Proof of Grahl-N1 and Grahl-N2

We can assume that $a = 1$. If not, take c such that $c^{1-n} = a$ and replace f, g by cf, cg, resp. and b by $b \cdot c^n$.
Then

$$(cf)^n + 1 \cdot (cf)^{(k)} = c^n f^n + c^n a f^{(k)} = c^n (f^n + a f^{(k)}) = c^n b$$

(if $f^n + af^{(k)} = b$).
Of course, ψ_f is non-constant since otherwise from the lemma on the logarithmic derivative we would obtain

$$\begin{aligned} n \cdot T(r,f) &= T\left(r, f^n\right) = T\left(r, f^{(k)}\right) + O(1) \\ &\leq T(r,f) + m\left(r, \frac{f^{(k)}}{f}\right) + k \cdot \overline{N}(r,f) + O(1) \\ &\leq (k+1) \cdot T(r,f) + S(r,f), \end{aligned}$$

which in view of $n \geq k+2$ would give $T(r,f) = S(r,f)$, a contradiction. For the same reason, ψ_g is non-constant, too.

Claim

$$f^{(k)} \equiv b \Rightarrow g^{(k)} \equiv b.$$

We assume by negation $g^{(k)} \not\equiv b$. If we apply Yi's extension of the Tumura-Clunie Theorem and Döringer's Lemma, we obtain

$$(n-3) \cdot T(r,g) \leq (k+1) \cdot \overline{N}(r,g) + S(r,g).$$

This gives a contradiction both in the meromorphic case (where $n-3 > k+1$) and in the entire case (where $\overline{N}(r,g) = 0$). So $g^{(k)} \equiv b$, and the claim is proved.

Now, since f^n and g^n share the value zero CM and since f and g have turned out to be polynomials, there exists some $\alpha \in \mathbb{C}$ such that $f = \alpha g$. From this and $f^{(k)} \equiv b \equiv g^{(k)} \neq 0$ we see that even $f \equiv g$. So the assertion of the theorem holds in this case.

The case that $g^{(k)} \equiv b$ can be treated in the same way.

From now on, we assume $f^{(k)} \not\equiv b$ and $g^{(k)} \not\equiv b$.

We define

$$\varphi_f := \frac{f^n}{\psi_f - b} \quad \text{and} \quad \varphi_g := \frac{g^n}{\psi_g - b}.$$

If φ_f would be constant, $\varphi_f \equiv c$, then we would have

$$f^n(1-c) \equiv c \cdot \left(f^{(k)} - b\right),$$

and we easily deduce a contradiction.
Therefore φ_f is not constant, and neither is φ_g. It is easy to show also that $S(r, \varphi_f) = S(r, f)$. Since φ_f is analytic at the poles of f, we have

$$\overline{N}(r, \varphi_f) \leq \overline{N}\left(r, \frac{1}{\psi_f - b}\right),$$

and from

$$\frac{1}{\varphi_f} = 1 + \frac{f^{(k)} - b}{f^n} \quad \text{and} \quad \frac{1}{\varphi_f - 1} = \frac{\psi_f - b}{b - f^{(k)}} = -1 - \frac{f^n}{f^{(k)} - b},$$

We apply the Second Fundamental Theorem to obtain

$$T(r,\varphi_f) \le \overline{N}\,(r,\varphi_f) + \overline{N}\left(r,\frac{1}{\varphi_f}\right) + \overline{N}\left(r,\frac{1}{\varphi_f - 1}\right) + S(r,f)$$
$$\le \overline{N}\left(r,\frac{1}{\psi_f - b}\right) + \frac{1}{n}\cdot N\left(r,\frac{1}{\varphi_f}\right) + \overline{N}\left(r,\frac{1}{f^{(k)} - b}\right) + \overline{N}(r,f) + S(r,f);$$

All terms except $\overline{N}\left(r,\frac{1}{\psi_f - b}\right)$ have upper bound of the form $CT(r,f)$ where C is independent of n.

The main difficulty will be to obtain an estimate for $\overline{N}\left(r,\frac{1}{\psi_f - b}\right)$, specifically, to obtain an estimate for the simple zeros of $\psi_f - b$.

Now consider a simple zero of z_0 of $\psi_f - b$ and hence of $\psi_g - b$. Then $\frac{\psi_f'}{\psi_f - b}$ has the Laurent expansion

$$\frac{\psi_f'}{\psi_f - b}(z) = \frac{\psi_f'(z_0) + \psi_f''(z_0)(z - z_0) + \dots}{\psi_f'(z_0)(z - z_0) + \frac{1}{2}\psi_f''(z_0)(z - z_0)^2 + \dots}$$
$$= \frac{1}{z - z_0} + \frac{1}{2} \cdot \frac{\psi_f''}{\psi_f'}(z_0) + \dots,$$

and similarly

$$\frac{\psi_g'}{\psi_g - b}(z) = \frac{\psi_g'(z_0) + \psi_g''(z_0)(z - z_0) + \dots}{\psi_g'(z_0)(z - z_0) + \frac{1}{2}\psi_g''(z_0)(z - z_0)^2 + \dots}$$
$$= \frac{1}{z - z_0} + \frac{1}{2} \cdot \frac{\psi_g''}{\psi_g'}(z_0) + \dots .$$

Thus, z_0 is a zero of $\tilde{H}(z) = \frac{\psi_f'}{\psi_f - b} - \frac{\psi_g'}{\psi_g - b} - \frac{1}{2}\left[\frac{\psi_f''}{\psi_f'} - \frac{\psi_g''}{\psi_g'}\right]$. We denote

$$D(z) := \frac{\psi_f'}{\psi_f - b}(z) - \frac{\psi_g'}{\psi_g - b}(z).$$

We have

$$\overline{N}_{1)}\left(r,\frac{1}{\psi_f - b}\right) \le \overline{N}\left(r,\frac{1}{\tilde{H}}\right) \le T(r,\tilde{H}) + O(1)$$
$$= m(r,\tilde{H}) + N(r,\tilde{H}) + O(1).$$

But here, one major problem occurs: $m(r,\widetilde{H})$ is small once more, but it seems that $N(r,\widetilde{H})$ cannot be controlled in the required way.

The solution to this problem is the following: If z_0 is a simple zero of $\psi_f - b$, then we use the equation $f^n(z_0) = b - f^{(k)}(z_0)$ to replace those terms which are "large" in the sense of Nevanlinna theory (i.e., with characteristic $n \cdot T(r,f)$) by smaller ones (with characteristic $\sim c \cdot T(r,f)$ where c is independent of n).

We obtain

$$\frac{\psi_f''}{\psi_f'}(z_0) = \frac{n(n-1)f^n f'^2 + nf^{n+1}f'' + f^2 f^{(k+2)}}{nf^{n+1}f' + f^2 f^{(k+1)}}(z_0)$$
$$= \frac{n(n-1)f'^2\left(b - f^{(k)}\right) + nff''\left(b - f^{(k)}\right) + f^2 f^{(k+2)}}{nff'\left(b - f^{(k)}\right) + f^2 f^{(k+1)}}(z_0).$$

Therefore, instead of $\widetilde{H}$ we introduce the more complicated auxiliary function

$$H := D - Q[f] + Q[g]$$

where

$$Q[f] := \frac{1}{2} \cdot \frac{n(n-1)f'^2\left(b - f^{(k)}\right) + nff''\left(b - f^{(k)}\right) + f^2 f^{(k+2)}}{f^2 f^{(k+1)} + nff'\left(b - f^{(k)}\right)}.$$

Then every simple zero of $\psi_f - b$ is a zero of H. The main advantage of H is that it no longer contains any terms involving f^n.

Moreover, also $Q[f]$, $Q[g]$ turn out to be a sum of logarithmic derivatives. Now the assumption that $H \not\equiv 0$ gives a contradiction. Therefore, we can assume that $H \equiv 0$, and this leads, after integration, to the desired connections between f and g.

A new formula for the natural logarithm of a natural number

Let $T \geq 2$ be an integer

$$\text{Ln}\, T = \lim_{x \to 1^-} Ln(1 + x + \cdots + x^{T-1}) = \lim_{x \to 1^-} \text{Ln}\, \frac{1 - x^T}{1 - x}$$

$$= \lim[\text{Ln}(1 - x^T) - \text{Ln}(1 - x)]$$

$$= \lim \left[- \left(x^T + \frac{x^{2T}}{2} + \frac{x^{3T}}{3} + \dots \right) + \left(x + \frac{x^2}{2} + \frac{x^3}{3} + \dots \right) \right]$$

$$= \lim \left[\left(\underbrace{x + \frac{x^2}{2} + \cdots + \frac{x^T}{T}} - x^T \right) \right.$$

$$+ \left(\underbrace{\frac{x^{T+1}}{T+1} + \frac{x^{T+2}}{T+2} + \cdots + \frac{x^{2T}}{2T}} - \frac{x^{2T}}{2} \right)$$

$$\left. + \left(\underbrace{\frac{x^{2T+1}}{2T+1} + \cdots + \frac{x^{3T}}{3T}} - \frac{x^{3T}}{3} \right) + \dots \right]$$

$$= \left(1 + \frac{1}{2} + \cdots + \frac{1}{T-1} - \frac{(T-1)}{T}\right)$$
$$+ \left(\frac{1}{T+1} + \cdots + \frac{1}{2T-1} - \frac{(T-1)}{2T}\right)$$
$$+ \left(\frac{1}{2T+1} + \cdots + \frac{1}{3T-1} - \frac{(T-1)}{T}\right) + \cdots$$

For example:

$$\operatorname{Ln} 2 = \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots$$
$$\operatorname{Ln} 3 = \left(\frac{1}{1} + \frac{1}{2} - \frac{2}{3}\right) + \left(\frac{1}{4} + \frac{1}{5} - \frac{2}{6}\right) + \cdots$$
$$\operatorname{Ln} 4 = \left(\frac{1}{1} + \frac{1}{2} + \frac{1}{3} - \frac{3}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} - \frac{1}{8}\right) + \cdots$$

THANK YOU!